

FLOPS AND MUTATIONS FOR CREPANT RESOLUTIONS OF POLYHEDRAL SINGULARITIES

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ABSTRACT. We prove that any crepant resolution of a polyhedral singularity \mathbb{C}^3/G for $G \subset SO(3)$ of types $\mathbb{Z}/n\mathbb{Z}$, D_{2n} and \mathbb{T} is isomorphic to a moduli space of representations of a quiver with relations. Moreover we classify all crepant resolutions explicitly by giving an open cover and find a one-to-one correspondence between them and mutations of the McKay quiver.

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1. INTRODUCTION

The general setup can be formulated as follows: let M be a quasiprojective variety with $K_M = 0$ and $G \subset \text{Aut}(X)$ a finite subgroup acting trivially on a global basis of $H^0(K_M)$. Then we can consider the diagram

$$\begin{array}{ccc} & M & \\ & \downarrow f & \\ Y & \xrightarrow{\pi} & X := M/G \end{array}$$

where Y is a resolution of singularities of X and $K_Y = \pi^*K_X + \sum a_i E_i$. The resolution π is said to be *crepant* if $K_Y = \pi^*K_X$, or in other words, if the discrepancy divisor $\sum a_i E_i$ is zero. We assume that $M = \mathbb{C}^3$ and the group G is a finite subgroup of $SL(3, \mathbb{C})$. This type of resolution had been known among algebraic geometers for a long time, although the relation between a crepant resolution Y and the representation theory of the group G was first realized by the work of the string theorists [DHVW] proposing the equality between the Euler number of Y and the number of conjugacy classes of G (or the number of irreducible representations of G). Their calculations were reformulated by Hirzebruch and Höfer [HH] into the Euler number conjecture, motivating a series

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of works by Bertin, Ito, Markushevich, Roan and others [BM94, Mar, Ito94, Ito95, Roa94, Roa96] finishing the verification of the formula with explicit methods case by case in dimension 3.

The relation with the so called McKay correspondence was further studied by Ito, Nakamura and Reid [Nak, IN99, Rei96, Rei02], and constitutes the initial motivation of our work. The central object in this approach is the so called G -Hilb or moduli space of G -clusters, which is always a crepant resolution of \mathbb{C}^3/G by the celebrated paper of Bridgeland, King and Reid [BKR].

It is well known from [IN] that we can interpret $G\text{-Hilb}(\mathbb{C}^3)$ as a moduli space \mathcal{M}_θ of θ -stable representations of the McKay quiver for a particular choice of θ in the space of stability conditions Θ . This point of view has lead to a substantial success starting from the earlier mentioned [BKR] where it was used to establish the McKay correspondence for $G \subset SL(3, \mathbb{C})$ as the equivalence of derived categories $D(Y) \cong D^G(\mathbb{C}^3)$. Moreover, the techniques in [BKR] apply to the generalized notion of G -cluster so that \mathcal{M}_C is a crepant resolution of \mathbb{C}^3/G and the equivalence $D(\mathcal{M}_C) \cong D^G(\mathbb{C}^3)$ also holds for the so called moduli space \mathcal{M}_C of G -constellations for any finite subgroup $G \subset SL(3, \mathbb{C})$ and any chamber $C \subset \Theta$ (cf. [CI]).

This paper focuses on the problem of describing every crepant resolution of \mathbb{C}^3/G for a given $G \subset SL(3, \mathbb{C})$ which has been treated much less in the literature. It was conjectured (and proved in the Abelian case) by Craw and Ishii in [CI] that every projective crepant resolution is isomorphic to a \mathcal{M}_C for some $C \subset \Theta$. In this paper we describe every crepant resolution of \mathbb{C}^3/G for $G \subset SO(3)$ of types $\mathbb{Z}/n\mathbb{Z}$, D_{2n} and \mathbb{T} by providing an open cover of each of them consisting of a finite number of copies of \mathbb{C}^3 (see Theorem 1.1 below). They are constructed as moduli spaces of θ -stable representations of the McKay quiver or G -constellations, verifying the Craw-Ishii conjecture for these family of groups. In addition, we study the chambers C for every crepant resolution X in Theorems 7.4, 7.9, and we conclude that there exists a connected region in Θ which contains all crepant resolutions of \mathbb{C}^3/G , where every wall crossing in that region corresponds to a flop (see Corollary 7.7).

We recall that $G\text{-Hilb}(\mathbb{C}^3)$ is the moduli space of G -clusters in \mathbb{C}^3 , where a G -cluster is a 0-dimensional subscheme $\mathcal{Z} \subset \mathbb{C}^3$ such that $\mathcal{O}_{\mathcal{Z}} \cong \mathbb{C}[G]$ the regular representation of G as $\mathbb{C}[G]$ -modules. For the finite subgroups $G \subset SO(3)$ the equivariant Hilbert scheme $G\text{-Hilb}(\mathbb{C}^3)$ was first studied by Gomi, Nakamura and Shinoda in [GNS1, GNS2] and constitutes the starting point of the calculations of this paper. From their work it is known that the fibre over the origin $\pi^{-1}(0)$ of $\pi : G\text{-Hilb}(\mathbb{C}^3) \rightarrow \mathbb{C}^3/G$ is 1-dimensional. This fact gives the one to one correspondence between smooth rational curves $E \subset \pi^{-1}(0)$ and nontrivial irreducible representations of G , providing the McKay correspondence for these groups (see also [BS07]).

In terms of the McKay quiver Q , there is a one to one correspondence between irreducible components of $\pi^{-1}(0)$ and vertices Q except the trivial vertex 0. Moreover, the dual graph of $\pi^{-1}(0)$ is precisely the graph of Q (i.e. forgetting the orientations of the arrows) removing $0 \in Q_0$.

On the other hand, non-commutative crepant resolutions (=NCCRs) are philosophically considered as a non-commutative analogue of crepant resolutions ([VdB, IW]). The Jacobian algebra $\mathcal{P}(Q, W)$ of the McKay quiver is an NCCR (see Section 3 in detail) and it corresponds to the crepant resolution isomorphic to $G\text{-Hilb}(\mathbb{C}^3)$ by the above correspondence where W is the potential provided by [BSW] (see Section 3). We therefore consider mutations $\mu(Q, W)$ of the McKay quiver (Q, W) with potential to obtain new NCCRs, which is a common operation in representation theory of algebras. In Section 5 we give an explicit description of all possibly mutations of the McKay quiver with potential under suitable rules, and we conclude that there is only a finite number of such mutations.

It turns out that by mutating at non-trivial vertex k without loops we match what is happening geometrically when flopping the curve $E_k \subset \pi^{-1}(0)$. In other words, for every group considered in this paper there is a one to one correspondence between flops from $G\text{-Hilb}(\mathbb{C}^3)$ and mutations of the McKay quiver Q with potential W .

Thus the main theorem of the paper is the following.

Theorem 1.1. Let $\pi : X \rightarrow \mathbb{C}^3/G$ be a crepant resolution. Then,

- (i) X is covered by a finite number of affine open sets U_i where $U_i \cong \mathbb{C}^3$.
- (ii) There exists a 1-to-1 correspondence between flops of $G\text{-Hilb}(\mathbb{C}^3)$ and mutations of (Q, W) .

The theorem is proved in Section 6.4. The open cover provided in Theorem 1.1 (i) for all crepant resolutions of the polyhedral singularities are obtained as moduli spaces of the McKay quiver Q for some chamber $C \in \Theta$. As mention before, this means that for this groups the Craw-Ishii conjecture holds.

Corollary 1.2. Let $G \subset SO(3)$ be a finite subgroup of type $\mathbb{Z}/n\mathbb{Z}$, D_{2n} or \mathbb{T} . Then every projective crepant resolution of \mathbb{C}^3/G is isomorphic to \mathcal{M}_C for some chamber $C \in \Theta$.

Moreover, for any crepant resolution $\pi : X \rightarrow \mathbb{C}^3/G$ we obtain the description of the dual graph of $\pi^{-1}(0)$ by looking at the graph of the corresponding mutated quiver μQ removing the trivial vertex. In addition, μQ also provides the information about the degrees of the normal bundles of every rational curve $E_i \subset \pi^{-1}(0)$ by the number of loops at the vertex i . Although there is no relation with irreducible representations, this extends the McKay correspondence for subgroups in $GL(2, \mathbb{C})$ of Wemyss [Wem08].

Corollary 1.3. Let $\pi : X \rightarrow \mathbb{C}^3/G$ be a crepant resolution and μQ the corresponding mutated quiver. Then the dual graph of $\pi^{-1}(0) \subset X$ is the same as the graph of μQ removing the trivial vertex. Moreover,

$$\begin{aligned} \{(-1, -1)\text{-curves in } X\} &\xleftrightarrow{1\text{-to-1}} \{\text{vertices in } \mu Q \text{ with no loops}\} \\ \{(-2, 0)\text{-curves in } X\} &\xleftrightarrow{1\text{-to-1}} \{\text{vertices in } \mu Q \text{ with one loop}\} \\ \{(-3, 1)\text{-curves in } X\} &\xleftrightarrow{1\text{-to-1}} \{\text{vertices in } \mu Q \text{ with two loops}\} \end{aligned}$$

In particular, this concludes that there is a one to one correspondence between crepant resolutions and NCCRs.

The way of finding the crepant resolution X in the corresponding mutation $\mu(Q, W)$ is shown in the following result (= Theorem 7.9), which states that X is the moduli space of representations of $\mu(Q, W)$ of dimension vector $\omega \mathbf{d}$ and the 0-generated stability condition θ^0 . Moreover it also provides the way of finding the parameter such that the moduli space of McKay quiver representations is isomorphic to X . The result was motivated by the work of [SY] in dimension 2 and uses methods in representation theory.

Theorem 1.4. Let $X \rightarrow \mathbb{C}^3/G$ be an arbitrary crepant resolution. Then $X \cong \mathcal{M}_{\theta^0, \omega \mathbf{d}}(\Gamma)$ for the corresponding Jacobian algebra $\Gamma = \mathcal{P}(\mu(Q, W))$. Moreover, there exists a corresponding sequence of wall crossings from $G\text{-Hilb}(\mathbb{C}^3)$ which leads to $X \cong \mathcal{M}_{\theta, \mathbf{d}}(\Lambda)$ where $\Lambda = \mathcal{P}(Q, W)$ and the chamber $C \in \Theta_{\mathbf{d}}$ containing θ is given by the inequalities $\theta(\omega^{-1} \mathbf{e}_i) > 0$ for any $i \neq 0$.

For a $G \subset SO(3)$ of type \mathbb{O} and \mathbb{I} , there exists a floppable $(-2, 0)$ or $(-3, 1)$ -curve in the fibre of origin of some crepant resolution of \mathbb{C}^3/G . This fact algebraically corresponds to the existence of a vertex i with loops such that $\dim_{\mathbb{C}} \Lambda_i < \infty$ in the corresponding mutated quiver (see Section 4 for details). In this case we do not know the appropriate definition of mutation of quivers with potentials, so we do not have an efficient method to calculate all mutations of the McKay quiver with potential. This problem will be treated in a future work.

The paper is organized as follows. After introducing the notations and conventions, in Section 2 we make a brief introduction to the finite subgroups of $SO(3)$ also called polyhedral groups. In Section 3 we describe the McKay quiver Q and the potential W using the [BSW] method for the polyhedral subgroups in $SO(3)$. Section 4 describes mutations of quiver with potentials and Section 5 calculates all possible mutations of the McKay quiver with potential (Q, W) for subgroups $G \subset SO(3)$. In Section 6 we describe explicitly every projective crepant resolution of \mathbb{C}^3/G with $G \subset SO(3)$ of types $\mathbb{Z}/n\mathbb{Z}$, D_{2n} and \mathbb{T} as moduli spaces \mathcal{M}_{θ} of representations of the

McKay quiver (Q, W) . Section 7 is dedicated to the space of stability conditions Θ for the moduli spaces \mathcal{M}_θ and the relation between changing the stability condition and mutating at a vertex $k \in Q$. In Section 8 we proof the Lemma which allows us to calculate explicitly every crepant resolution by flopping only at $(-1, -1)$ -curves, and finally the Appendix shows the representation spaces for every open set in any crepant resolution of \mathbb{C}^3/G for $G \subset SO(3)$ of types $\mathbb{Z}/n\mathbb{Z}$, D_{2n} and \mathbb{T} .

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1.1. Notations and conventions. We refer to [ASS] for notations on quivers and their representations, [IN99] for G -Hilbert schemes, [Rei02] for the McKay correspondence and [Wem, King] for moduli spaces of quiver representations. We always take \mathbb{C} as ground field although everything can be done in any algebraically closed field of characteristic 0.

Abusing the notation, we indistinguishably use vertices in a quiver Q and their corresponding vector spaces in a representations of Q . If in addition Q is the McKay quiver we also treat them as irreducible representations of Q .

The notion of stability of a representation of Q is defined as follows (cf. [King]): let M be a representation of Q of dimension vector $\mathbf{d} = (d_i)_{i \in Q_0}$, let $\theta \in \mathbb{Q}^{Q_0}$ and define $\theta(M) := \sum \theta_i d_i$. Then M is θ -(semi)stable if $\theta(M') > 0 = \theta(M)$ for $0 \subsetneq M' \subsetneq M$ (with the usual \geq for semistability). Given $\mathbf{d} \in \mathbb{Q}^{Q_0}$ we denote the space of stability conditions by $\Theta_{\mathbf{d}} := \{\theta \in \mathbb{Q}^{Q_0} | \theta \cdot \mathbf{d} = 0\} \subset \Theta \cong \mathbb{Q}^{Q_0}$. The stability parameter $\theta \in \Theta_{\mathbf{d}}$ is said to be *generic* if every θ -semistable representation is θ -stable. We only consider the space of generic stability conditions $\Theta_{\mathbf{d}}^{\text{gen}}$ which forms an open and dense subset in $\Theta_{\mathbf{d}}$.

Since we consider moduli spaces of modules over path algebras $\Lambda := \mathbb{C}Q/R$ for different quivers, we denote by $\mathcal{M}_{\theta, \mathbf{d}}(\Lambda)$ the moduli space of θ -stable representations of the quiver Q with relations R of dimension vector \mathbf{d} . If (Q, R) is the McKay quiver of G with relations R as in [BSW] and $\mathbf{d} := (\dim V)_{V \in \text{Irr } G}$, then we denote $\mathcal{M}_{\theta, \mathbf{d}}(\Lambda)$ simply by \mathcal{M}_θ or \mathcal{M}_C for some $\theta \in C$ where C is a chamber in $\Theta_{\mathbf{d}}$.

Given a quiver with potential (Q, W) we denote by $\mathcal{P}(Q, W)$ the Jacobian algebra and by $\mu(Q, W)$ the quiver with potential obtained by a sequence of mutations or simply μQ if the potential is clear by the context.

Since the we use GIT methods, by crepant resolution $\pi : Y \rightarrow X$ we always mean projective.

2. FINITE SUBGROUPS OF $SO(3)$

Let G be a finite subgroup of the special orthogonal group $SO(3)$ which consists of rotations about $0 \in \mathbb{R}^3$. These groups are the so called *polyhedral groups* and classified into five cases: cyclic, dihedral, tetrahedral, octahedral and icosahedral (see Table 1).

polyhedral group	isomorphic group	order
cyclic	$\mathbb{Z}/n\mathbb{Z}$	n
dihedral D_{2n}	$\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$	$2n$
tetrahedral \mathbb{T}	A_4	12
octahedral \mathbb{O}	S_4	24
icosahedral \mathbb{I}	A_5	60

TABLE 1. Finite subgroups of $SO(3)$

2.1. The cyclic group of order $n + 1$. Let G be the cyclic subgroup of $SO(3)$ of order $n + 1$. Then G is of the form:

$$G = \langle \sigma = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle, \text{ where } \epsilon = e^{2\pi \frac{i}{n+1}}.$$

We also denote it by $G = \frac{1}{n+1}(1, n, 0)$. The character table of G is given in Table 2.

conjugacy classes	1	σ	\dots	σ^i	\dots	σ^n
number of c.c	1	1	\dots	1	\dots	1
V_0	1	1	\dots	1	\dots	1
\vdots	\vdots	\vdots		\vdots		\vdots
V_j	1	ϵ^j	\dots	ϵ^{ij}	\dots	ϵ^{jn}
\vdots	\vdots	\vdots		\vdots		\vdots
V_n	1	ϵ^n	\dots	ϵ^{in}	\dots	ϵ^{n^2}

TABLE 2. Characters of G of type $\mathbb{Z}/n\mathbb{Z}$.

2.2. The dihedral group of order $2n$. Let n be a positive integer and G be the dihedral subgroup D_{2n} of $SO(3)$ of order $2n$. Then G is generated by:

$$G = \langle \sigma = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon^{n-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle, \text{ where } \epsilon = e^{2\pi i/n}.$$

These groups are divided into two cases, depending on the parity of n . For the case $n = 2m$ even, the group has four 1-dimensional irreducible representations $V_0, V_{0'}, V_m$ and $V_{m'}$, and $m - 1$ 2-dimensional irreducible representations V_j for $1 \leq j \leq m - 1$. The character table is given in Table 3.

c.c	1	-1	τ	$\tau\sigma$	σ^i
# of c.c	1	1	m	m	2
V_0	1	1	1	1	1
$V_{0'}$	1	1	-1	-1	1
V_m	1	$(-1)^m$	1	-1	$(-1)^i$
$V_{m'}$	1	$(-1)^m$	-1	1	$(-1)^i$

TABLE 3. Characters of D_{2n} , with $n = 2m$ even and $1 \leq i, j \leq m - 1$.

For the case $n = 2m + 1$ odd, the group has two 1-dimensional representations V_0 and $V_{0'}$, and m 2-dimensional representations V_j for $1 \leq j \leq m$. The character table is given in Table 4.

c.c	1	τ	σ^i
# of c.c	1	$2m + 1$	2
V_0	1	1	1
$V_{0'}$	1	-1	1
V_j	2	0	$\epsilon^{ij} + \epsilon^{-ij}$

TABLE 4. Characters of D_{2n} , with $n = 2m + 1$ odd and $1 \leq i, j \leq m$.

In both cases, the representations V_j are realized as $V_j(\sigma) = \begin{pmatrix} \epsilon^j & 0 \\ 0 & \epsilon^{-j} \end{pmatrix}, V_j(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

2.3. The tetrahedral group. Let G be the tetrahedral subgroup \mathbb{T} of $SO(3)$. Then

$$G = \langle \sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rangle.$$

The group G is isomorphic to the alternating group A_4 and it has order 12. This group is also called *trihedral group of order 12* and the character table of G is shown in Table 5.

c.c	1	σ	τ	τ^2
# of c.c	1	3	4	4
V_0	1	1	1	1
V_1	1	1	ω	ω^2
V_2	1	1	ω^2	ω
V_3	3	-1	0	0

TABLE 5. Characters of tetrahedral group of order 12.

2.4. The octahedral group. Let G be the octahedral subgroup \mathbb{O} of $SO(3)$. Then:

$$G = \langle \sigma = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rangle.$$

The group G is isomorphic to the symmetric group S_4 and its order is 24. The character table of G is given in Table 6.

c.c	1	σ^2	τ	σ	$\sigma\tau\sigma^2$
# of c.c	1	3	8	6	6
V_0	1	1	1	1	1
V_1	1	1	1	-1	-1
V_2	2	2	-1	0	0
V_3	3	-1	0	1	-1
V_4	3	-1	0	-1	1

TABLE 6. Characters of octahedral group

The representation V_3 is the representation given the inclusion G to $SO(3)$, i.e. the natural representation. The irreducible representations V_2 and V_4 are realized as $V_2(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $V_2(\tau) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$, $V_4(\sigma) = -\sigma$, $V_4(\tau) = \tau$.

2.5. The icosahedral group. Let G be the icosahedral subgroup \mathbb{I} of $SO(3)$:

$$G = \langle \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon^4 \end{pmatrix}, \tau = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix} \rangle,$$

where $\epsilon = e^{2\pi i/5}$, $s = \epsilon^2 + \epsilon^3 = \frac{-1-\sqrt{5}}{2}$ and $t = \epsilon + \epsilon^4 = \frac{-1+\sqrt{5}}{2}$. Note that $v = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \sigma^3\tau$ (compare [GNS2, 3.1] and [YY]). The group G is isomorphic to the alternating group A_5 and its order is 60. The character table of G is given in Table 7.

The natural representation is V_1 , the irreducible representation V_2 is realized as

$$V_2(\sigma) = \sigma^2, V_2(\tau) = \tau.$$

c.c	1	$\sigma\tau$	τ	σ	σ^2
# of c.c	1	20	15	12	12
V_0	1	1	1	1	1
V_1	3	0	-1	$-s$	$-t$
V_2	3	0	-1	$-t$	$-s$
V_3	4	1	0	-1	-1
V_4	5	-1	1	0	0

TABLE 7. Characters of icosahedral group

and the 4-dimensional irreducible representation V_3 is obtained by removing the unit representation from the permutation representation of A_5 on $\{a, b, c, d, e\}$. If we take a suitable basis of V_3 , it is realized as follows:

$$V_3(\sigma) = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon^2 & 0 & 0 \\ 0 & 0 & \epsilon^3 & 0 \\ 0 & 0 & 0 & \epsilon^4 \end{pmatrix}, V_3(\tau) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & t & -s & -1 \\ t & -1 & 1 & -s \\ -s & 1 & -1 & t \\ -1 & -s & t & 1 \end{pmatrix}.$$

The 5-dimensional irreducible representation V_4 is a representation obtained by removing the unit representation from the permutation representation of A_5 on the set of its 6 subgroups of order 5. Taking a suitable basis, it is realized as follows:

$$V_4(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 & 0 \\ 0 & 0 & \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 & \epsilon^3 & 0 \\ 0 & 0 & 0 & 0 & \epsilon^4 \end{pmatrix}, V_4(\tau) = \frac{1}{5} \begin{pmatrix} -1 & -6 & -6 & -6 & -6 \\ -1 & 1-t & 2s & 2t & 1-s \\ -1 & 2s & 1-s & 1-t & 2t \\ -1 & 2t & 1-t & 1-s & 2s \\ -1 & 1-s & 2t & 2s & 1-t \end{pmatrix}.$$

3. MCKAY QUIVER WITH POTENTIAL FOR $G \subset SO(3)$

First we recall quivers with potentials. Let Q be an arbitrary finite connected quiver (possibly with loops and 2-cycles) with a vertex set Q_0 and an arrow set Q_1 . For an arrow a , ha and ta denote the head and tail of a respectively. We denote by $\mathbb{C}Q_i$ the \mathbb{C} -vector space with basis Q_i consisting of paths of length i in Q , and by $\mathbb{C}Q_{i,\text{cyc}}$ the subspace of $\mathbb{C}Q_i$ spanned by all cycles. A *quiver with potential* (QP for short) is a pair (Q, W) consisting of Q and an element $W \in \bigoplus_{i \geq 2} \mathbb{C}Q_{i,\text{cyc}}$ (called a *potential*). For an arrow $a \in Q_1$, the cyclic derivative $\partial_a W$ is defined by $\partial_a(a_1 \cdots a_\ell) = \sum_{a_i=a} a_{i+1} \cdots a_\ell a_1 \cdots a_{i-1}$ and extended linearly. The *Jacobian algebra* of a QP (Q, W) is defined by

$$\mathcal{P}(Q, W) := \mathbb{C}Q / \langle \partial_a W \mid a \in Q_1 \rangle.$$

In the rest of this section, for a finite subgroup G of $SO(3)$, we give an explicit description of the McKay quiver with potential (McKay QP for short) whose Jacobian algebra is Morita equivalent to the skew group ring $S * G$ by using the method in [BSW]. Since G is embedded in $SO(3)$, G acts on $V = \mathbb{C}^3$ naturally and dually on the polynomial ring $S = \mathbb{C}[V]$. The skew group ring $S * G$ is a free S -module $S \otimes_{\mathbb{C}} G$ with basis G with multiplications given by $(s \otimes g)(s' \otimes g') = sg(s') \otimes gg'$ for any $s, s' \in S$ and $g, g' \in G$. The skew group ring is important since it is a non-commutative version of a crepant resolution.

The McKay quiver of G is a quiver Q such that the vertex set is the irreducible representations V_i of G and we draw a_{ij} arrows from V_i to V_j where $a_{ij} = \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}G}(V_i, V^* \otimes V_j) = \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}G}(V_j, V \otimes V_i)$. For simplicity we write $Q_0 = \{0, 1, \dots, n\}$ where $i \in Q_0$ corresponds to the irreducible representation V_i and in particular 0 the trivial representation.

We take a standard basis v_1, v_2 and v_3 of V . Note that $\text{Hom}_{\mathbb{C}G}(V_i, V^* \otimes V_j)$ is isomorphic to $\text{Hom}_{\mathbb{C}G}(V_j, V \otimes V_i)$ as a \mathbb{C} -vector space. For each arrow $a \in Q_1$ we consider a G -equivariant homomorphism $\varphi_a : V_{t(a)} \rightarrow V \otimes V_{h(a)}$. If $p = abc$ is a closed path of length 3, then by Schur's lemma, the composition of maps

$$V_{t(a)} \xrightarrow{\varphi_a} V^* \otimes V_{t(b)} \xrightarrow{id_V \otimes \varphi_b} V^{\otimes 2} \otimes V_{t(c)} \xrightarrow{id_{V^{\otimes 2}} \otimes \varphi_c} V^{\otimes 3} \otimes V_{h(c)} \xrightarrow{\alpha \otimes id_{V_{h(c)}}} \bigwedge^3 V \otimes V_{h(c)} \xrightarrow{\sim} V_{h(c)=t(a)}$$

is a constant (denoted by c_p), $\alpha : V^{\otimes 3} \rightarrow \bigwedge^3 V$ is the antisymmetrizer and the last map is an isomorphism given by the composition $\bigwedge^3 V \rightarrow V_0$; $v_1 \wedge v_2 \wedge v_3 \rightarrow \ell_0$ and $V_0 \otimes V_{h(c)} \rightarrow V_{h(c)}$; $\ell_0 \otimes v \rightarrow v$. Note that since $SO(3) \subset SL(3, \mathbb{C})$ we have that $\bigwedge^3 V$ is isomorphic to the trivial representation.

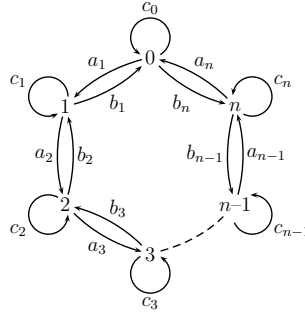
Theorem 3.1 ([BSW, Theorem 3.2]). If we take

$$W = \sum_{p=|3|} c_p (\dim V_{h(p)}) p$$

then $\mathcal{P}(Q, W)$ is Morita equivalent to $S * G$.

Remark 3.2. In [BSW], the above result is described by using superpotentials. But in the three dimensional case, superpotentials is nothing but potentials.

3.1. The cyclic group of order $n + 1$. Let G be a finite cyclic subgroup of $SO(3)$ of order $n + 1$. The McKay quiver Q of G is as follows:



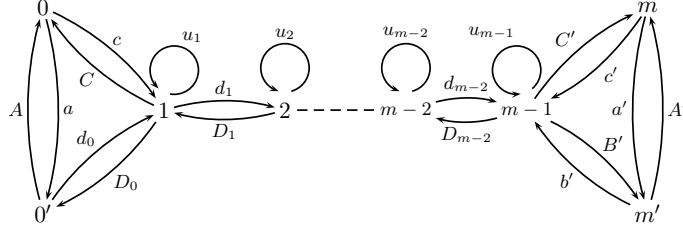
For each arrow, the corresponding G -equivariant homomorphism is given by

$$\begin{aligned} \varphi_{a_i} : V_i &\rightarrow V \otimes V_{i+1}; \ell_i \mapsto v_2 \otimes \ell_{i+1}, \\ \varphi_{b_i} : V_{i+1} &\rightarrow V \otimes V_i; \ell_{i+1} \mapsto v_1 \otimes \ell_i, \\ \varphi_{c_i} : V_i &\rightarrow V \otimes V_i; \ell_i \mapsto v_3 \otimes \ell_i \end{aligned}$$

where ℓ_i denotes a basis of V_i for any $i = 0, \dots, n$. In the following, for simplicity we just describe φ_{a_i} as v_2 etc. For any $i = 0, \dots, n$, we can check that $c_p = -1$ if $p = c_i a_i b_i$ and $c_p = 1$ if $p = c_i b_{i-1} a_{i-1}$, where $a_{-1} = a_{n-1}$ and $b_{-1} = b_n$. By definition of c_p , for all other 3-cycles p we have $c_p = 0$. Hence the McKay potential is given by

$$W = - \sum_{i=0}^n a_i b_i c_i + \sum_{i=0}^n b_{i-1} a_{i-1} c_i.$$

3.2. The dihedral group of order $2n$ (n even). Let G be a dihedral group D_{2n} where $n = 2m$ for some positive integer m . The McKay quiver Q of G is as follows:



and the corresponding G -equivariant maps are:

$$A, a, A', a' : v_3, d_0 : (v_2, -v_1), D_0 : \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}, c : (v_2, v_1), C : \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, b' : (-v_1, -v_2), B' : \begin{pmatrix} -v_2 \\ -v_1 \end{pmatrix},$$

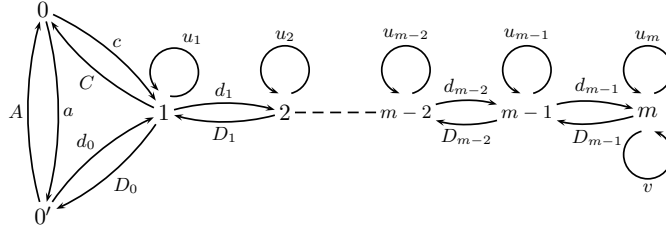
$$c' : (v_1, v_2), C' : \begin{pmatrix} v_2 \\ v_1 \end{pmatrix}, d_i : \begin{pmatrix} v_2 & 0 \\ 0 & v_1 \end{pmatrix} \quad (0 \leq i \leq m-2), D_i : \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \quad (0 \leq i \leq m-2),$$

$$u_i : \begin{pmatrix} v_3 & 0 \\ 0 & -v_3 \end{pmatrix} \quad (0 \leq i \leq m-1).$$

For example, $d_0 : (v_2, -v_1)$ means the G -equivariant map $\varphi_{d_0} : V_{0'} \rightarrow V \otimes V_1; \ell_{0'} \mapsto v_2 \otimes \ell_1^1 - v_1 \otimes \ell_1^2$ where $\ell_{0'}$ is the basis of $V_{0'}$ and ℓ_1^1, ℓ_1^2 is the basis of V_1 given in the previous section. Note that the above description depends on the choice of bases of V_i 's. For the above equivariant maps, one can calculate the superpotential:

$$W/2 = -ad_0C - cD_0A + u_1D_0d_0 + u_1Cc - \sum_{i=1}^{m-2} u_i d_i D_i + \sum_{i=2}^{m-1} u_i D_{i-1} d_{i-1} \\ - u_{m-1} B' b' - u_{m-1} C' c' - a' b' C' - c' B' A'.$$

3.3. The dihedral group of order $2n$ (n odd). Let G be a dihedral group D_{2n} where $n = 2m + 1$ for some positive integer m . The McKay quiver Q of G is as follows:



and the corresponding G -equivariant maps are:

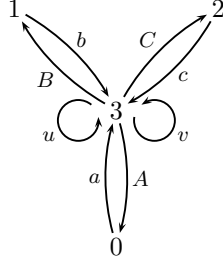
$$A, a : v_3, d_0 : (v_2, -v_1), D_0 : \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}, c : (v_2, v_1), C : \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, d_i : \begin{pmatrix} v_2 & 0 \\ 0 & v_1 \end{pmatrix} \quad (0 \leq i \leq m-1),$$

$$D_i : \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \quad (0 \leq i \leq m-1), u_i : \begin{pmatrix} v_3 & 0 \\ 0 & -v_3 \end{pmatrix} \quad (0 \leq i \leq m), v : \begin{pmatrix} 0 & v_2 \\ v_1 & 0 \end{pmatrix}.$$

For the above equivariant maps, the superpotential is given by

$$W/2 = -ad_0C - cD_0A + u_1D_0d_0 + u_1Cc - \sum_{i=1}^{m-1} u_i d_i D_i + \sum_{i=2}^m u_i D_{i-1} d_{i-1} - u_m v^2.$$

3.4. The tetrahedral group. Let G be the tetrahedral group of order 12. The McKay quiver Q of G is as follows.



and the corresponding G -equivariant maps are:

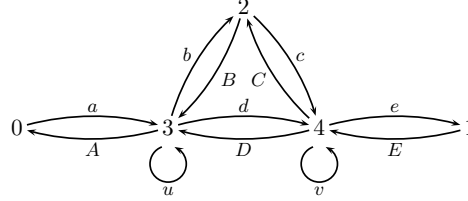
$$a : (v_1, v_2, v_3), A : \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, b : (v_1, \omega v_2, \omega^2 v_3), B : \begin{pmatrix} v_1 \\ \omega^2 v_2 \\ \omega v_3 \end{pmatrix}, c : (v_1, \omega^2 v_2, \omega v_3), C : \begin{pmatrix} v_1 \\ \omega v_2 \\ \omega^2 v_3 \end{pmatrix},$$

$$u : \begin{pmatrix} 0 & 0 & v_2 \\ v_3 & 0 & 0 \\ 0 & v_1 & 0 \end{pmatrix}, v : \begin{pmatrix} 0 & v_3 & 0 \\ 0 & 0 & v_1 \\ v_2 & 0 & 0 \end{pmatrix}.$$

For the above equivariant maps, the superpotential is given by

$$W/3 = uAa + \omega uBb + \omega^2 uCc - \frac{1}{3}u^3 - vAa - \omega^2 vBb - \omega vCc + \frac{1}{3}v^3.$$

3.5. The octahedral group. Let G be the tetrahedral group. The McKay quiver Q of G is the following:



and the G -equivariant maps are:

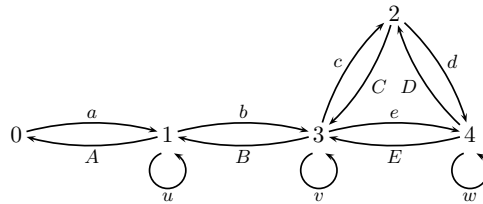
$$a, E : (v_1, v_2, v_3), A, e : \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, b : \begin{pmatrix} v_1 & \omega v_1 \\ \omega v_2 & v_2 \\ \omega^2 v_3 & v_3 \end{pmatrix}, B : \begin{pmatrix} v_1 & \omega^2 v_2 & \omega v_3 \\ \omega^2 v_1 & v_2 & \omega v_3 \end{pmatrix}, c : \begin{pmatrix} v_1 & \omega^2 v_2 & \omega v_3 \\ -\omega^2 v_1 & -v_2 & -\omega v_3 \end{pmatrix},$$

$$C : \begin{pmatrix} v_1 & -\omega v_1 \\ \omega v_2 & -v_2 \\ \omega^2 v_3 & -v_3 \end{pmatrix}, d, D : \begin{pmatrix} 0 & v_3 & v_2 \\ v_3 & 0 & v_1 \\ v_2 & v_1 & 0 \end{pmatrix}, u, v : \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$$

For the above equivariant maps, the superpotential is given by

$$W/6 = uAa - ubB - udD - 1/3u^3 + veE - vCc - vDd + 1/3v^3 + (w^2 - w)dDB + (w^2 - w)Dbc.$$

3.6. The icosahedral group. Let G be the tetrahedral group. The McKay quiver Q of G is as follows:



and the G -equivariant maps are:

$$a : (2v_1, v_3, v_2), A : \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, u : \begin{pmatrix} 0 & v_3 & -v_2 \\ 2v_2 & -2v_1 & 0 \\ -2v_3 & 0 & 2v_1 \end{pmatrix}, b : \begin{pmatrix} -2v_1 & 3v_3 & 0 & 0 & 3v_2 \\ v_2 & 6v_1 & 6v_3 & 0 & 0 \\ v_3 & 0 & 0 & 6v_2 & 6v_1 \end{pmatrix},$$

$$\begin{aligned}
B : \begin{pmatrix} -4v_1 & v_3 & v_2 \\ v_2 & v_1 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \\ v_3 & 0 & v_1 \end{pmatrix}, v : \begin{pmatrix} 0 & 6v_3 & 0 & 0 & -6v_2 \\ v_2 & 2v_1 & -2v_3 & 0 & 0 \\ 0 & -2v_2 & 4v_1 & 0 & 0 \\ 0 & 0 & 0 & -4v_1 & 2v_3 \\ -v_3 & 0 & 0 & 2v_2 & -2v_1 \end{pmatrix}, c : \begin{pmatrix} 6v_1 & 0 & 0 \\ v_2 & v_3 & 0 \\ 0 & v_1 & v_3 \\ 0 & v_2 & v_1 \\ v_3 & 0 & v_2 \end{pmatrix}, \\
C : \begin{pmatrix} v_1 & v_3 & 0 & 0 & v_2 \\ 0 & 2v_2 & 2v_1 & 2v_3 & 0 \\ 0 & 0 & 2v_2 & 2v_1 & 2v_3 \end{pmatrix}, d : \begin{pmatrix} v_3 & 0 & 0 & v_2 \\ -v_2 & -v_1 & v_3 & 0 \\ 0 & v_2 & -2v_1 & -v_3 \end{pmatrix}, D : \begin{pmatrix} -v_2 & v_3 & 0 \\ 0 & 2v_1 & -v_3 \\ 0 & -v_2 & 2v_1 \\ -2v_3 & 0 & v_2 \end{pmatrix}, \\
e : \begin{pmatrix} 6v_3 & 0 & 0 & -6v_2 \\ -4v_1 & 2v_3 & 0 & 0 \\ v_2 & 2v_1 & 3v_3 & 0 \\ 0 & -3v_2 & -2v_1 & -v_3 \\ 0 & 0 & -2v_2 & 4v_1 \end{pmatrix}, E : \begin{pmatrix} v_2 & -4v_1 & v_3 & 0 & 0 \\ 0 & 2v_2 & 2v_1 & -3v_3 & 0 \\ 0 & 0 & 3v_2 & -2v_1 & -2v_3 \\ -v_3 & 0 & 0 & -v_2 & 4v_1 \end{pmatrix}, w : \begin{pmatrix} v_1 & v_3 & 0 & 0 \\ v_2 & -v_1 & 0 & 0 \\ 0 & 0 & v_1 & -v_3 \\ 0 & 0 & -v_2 & -v_1 \end{pmatrix}.
\end{aligned}$$

For the above equivariant maps, the superpotential is given by

$$W/12 = -uAa + 5ubB - \frac{2}{3}u^3 + 15vBb - 5vcC - 20vEe + \frac{10}{3}v^3 + 5weE - wDd - \frac{1}{3}w^3 - 5dec + 10CED$$

4. MUTATIONS OF QUIVERS WITH POTENTIALS AND NCCRS

In this section we define mutations of quivers with potentials. For a finite subgroup $G \subset SO(3)$ let $R = S^G$ be the invariant subring of the polynomial ring S . The skew group ring $S * G$ is a non-commutative crepant resolution (NCCR for short) of R ([IW, VdB]). Since NCCRs are closed under Morita equivalence by [IW, Lemma 2.11], the Jacobian algebra of the McKay QP is also an NCCR of R . To obtain new NCCRs from the McKay QP, we use mutation which is a standard technique in representation theory. The Jacobian algebra of the mutation is also an NCCR.

A QP (Q, W) is called *reduced* if $W \in \bigoplus_{i \geq 3} \mathbb{C}Q_{i, \text{cyc}}$. For a non-reduced QP (Q, W) , if there is a reduced QP (Q', W') such that $\mathcal{P}(Q, W) \simeq \mathcal{P}(Q', W')$, then we say that (Q', W') is the reduced part of (Q, W) .

For a QP (Q, W) let k be a vertex with no loops (but possibly lying on a 2-cycle). Then we define the *mutation* $\tilde{\mu}_k(Q, W)$ of (Q, W) in the following way:

- (1) Let Q' be a quiver obtained from Q as follows.
 - (a) Replace the fixed k in Q by a new vertex k^* .
 - (b) Add new arrows $[ab] : i \rightarrow j$ for each pair of arrows $a : i \rightarrow k$ and $b : k \rightarrow j$ in Q .
 - (c) Replace each arrow $a : i \rightarrow k$ in Q by a new arrow $a^* : k^* \rightarrow i$.
 - (d) Replace each arrow $b : k \rightarrow j$ in Q by a new arrow $b^* : j \rightarrow k^*$.
- (2) Let $W' = [W] + \Delta$ where $[W]$ and Δ are the following.
 - (a) $[W]$ is obtained by substituting $[ab]$ for each factor ab in W with $a : i \rightarrow k$ and $b : k \rightarrow j$.
 - (b) $\Delta = \sum_{a, b \in Q_1, ha=k=tb} [ab]b^*a^*$.

Moreover let $\mu_k(Q, W)$ be a reduced part of $\tilde{\mu}_k(Q, W)$ if it exists.

The important difference from the original mutation from [DWZ] is that $\mu_k(Q, W)$ may have loops and 2-cycles. However this situation is quite natural in some geometric contexts as we will see later.

Remark 4.1. We give a remark about the connection between our definition of QP and tilting mutation. Let $\Lambda = \mathcal{P}(Q, W)$ be a NCCR and P_k a projective right Λ -module associated to the vertex k . Let $f : P_k \rightarrow X$ be a left add Λ/P_k -approximation and put $K_k := \text{Coker } f$. If f is injective, then one can prove that $\Lambda/P_k \oplus K_k$ is a tilting Λ -module and $\text{End}_\Lambda(\Lambda/P_k \oplus K_k)$ is also a NCCR. By a similar strategy of [BIRS], we can prove that if a QP (Q, W) is gradable

(see below) and Λ is a 3-Calabi-Yau algebra, a tilting mutation coincides with a mutation of QP, that is, we have an isomorphism $\text{End}_\Lambda(\mu_k \Lambda) \simeq \mathcal{P}(\mu_k(Q, W))$. In our case, we can check that all mutations $\mu(Q, W)$ obtained from the McKay QP (Q, W) are gradable and $\mathcal{P}(\mu(Q, W))$ are 3-Calabi-Yau.

We do not know if the reduced part exists in general, however if there is a grading on the QP, the reduced part always exists. Given (Q, W) a QP, we can assign to any element $a \in Q_1$ a non-negative integer $\deg a$. For an element e in Q_0 we put $\deg e = 0$, and for a path $p = a_1 \cdots a_\ell$ in Q we put $\deg p = \sum_{i=1}^\ell \deg a_i$. We say that the potential W is *homogeneous* of degree d if all terms in W are of degree d .

Definition 4.2. We say a QP (Q, W) is gradable if there exist a grading of Q such that W is homogeneous of degree d .

Trivially the Jacobian algebra of a gradable QP becomes a graded algebra.

Lemma 4.3. If a QP (Q, W) is gradable, then there exists a reduced QP $(Q_{\text{red}}, W_{\text{red}})$ such that $\mathcal{P}(Q, W) \simeq \mathcal{P}(Q_{\text{red}}, W_{\text{red}})$.

Proof. If (Q, W) is reduced, we have nothing to do. So we assume that there is a 2-cycle ab which appears in W . Write $W = c_{ab}ab + W'$ where $c_{ab} \in \mathbb{C}$ is a non-zero element. Then $\partial_a W = c_{ab}b + \partial_a W'$ and $\partial_b W = c_{ab}a + \partial_b W'$. Without loss of generality we can assume that $\deg a \geq \deg b$. Then $\deg \partial_a W' = \deg b$, so $\partial_a W'$ does not contain a or b . Thus (Q, W) reduce to a QP whose potential does not contain a or b and Jacobian algebras are isomorphic. Of course, the new QP obtained is graded. By repeating this operation, we obtain the required reduced QP. \square

5. MUTATIONS OF MCKAY QUIVERS WITH POTENTIALS FOR $G \subset SO(3)$

Let $G \subset SO(3)$ be a finite subgroup of type $\mathbb{Z}/n\mathbb{Z}$, D_{2n} or \mathbb{T} . In this section we give the mutations of the McKay QP of G . We denote by \star the vertex 0 corresponding trivial representation of G . We consider mutations of QPs according to the following two rules:

Rules of the game.

- (1) We do not mutate at the vertex \star .
- (2) We do not mutate at a vertex with loops.

By this rule only finite many number of mutations appear. We explain where the rules of the game come from. For the rule (2) we recall the next result.

Theorem 5.1 ([IW]). Suppose R is a complete local normal three-dimensional Gorenstein ring with maximal modifying module M . Denote $A = \text{End}_R(M)$, let M_i be an indecomposable summand of M and consider $A_i := A/A(1 - e_i)A$ where e_i is the idempotent in A corresponding to M_i . Then if $\dim A_i = \infty$ then $\mu_k A \simeq A$.

Let (Q, W) be the McKay QP and $\Lambda = \mathcal{P}(Q, W)$. Then Λ is Morita equivalent to $S * G \simeq \text{End}_R(S)$ and we can decompose S to the direct sum of CM R -modules M_i . By taking non-isomorphic CM R -modules M_i , we have an isomorphism $\Lambda \simeq \text{End}_R(\bigoplus M_i)$. So, since $R = S^G$ is a normal three-dimensional Gorenstein ring, the above theorem can be applied to our case if we take a completion \widehat{R} . We will prove that, for any algebra Γ obtained from Λ by iterating mutations, $\dim_{\mathbb{C}} \Gamma_i = \infty$ for any vertex i with loops. Hence the rule (2) is valid. For subgroups $G \subset SO(3)$ of types $\mathbb{Z}/n\mathbb{Z}$, D_{2n} and \mathbb{T} , the rule (2) geometrically means that $(-2, 0)$ and $(-3, 1)$ -curves are not floppable in any crepant resolution of \mathbb{C}^3/G . However, for a $G \subset SO(3)$ of type \mathbb{O} and \mathbb{I} , there is a vertex $i \in Q$ with loops such that $\dim_{\mathbb{C}} \Lambda_i < \infty$ and in this case we do not know the appropriate definition of mutation of QPs.

We set the rule (1) because the vertex \star corresponds to the trivial representation, so it does not corresponds to any exceptional curve in the fibre of the origin. So geometrically it seems to be that there is no sense to mutate at the vertex \star . On the other hand, in the context in NCCRs we can say the following. By the definition, a NCCR Γ is of the form $\text{End}_R(M)$ for some reflexive module M . The rule (1) implies that M always contains R as a direct summand, equivalently M is a CM R -module, hence a CT module.

5.1. The cyclic group of order $n + 1$. Let (Q, W) be the McKay QP of G obtained in 3.1. Then by the rules of the game there is nothing to do, although we check that $\dim_{\mathbb{C}} \Lambda_i = \infty$ for confirmation. Indeed, for each vertex i , we have $\Lambda_i = \bigoplus_{\ell \geq 0} \mathbb{C} c_i^\ell$ thus infinite dimensional.

5.2. The dihedral group of order $2n$ (n even). Let $n = 2m$ with $m \geq 2$ and (Q, W) be the McKay QP of G obtained in 3.2. In this case there are $(m + 1)^2$ non-equivalent mutation QPs which are equivalent to (Q, W) .

Example 5.2 (The case $n = 6$). Before treating the general case we observe the case $n = 6$. In this case there are 16 non-equivalent QPs, see the Figure 1.

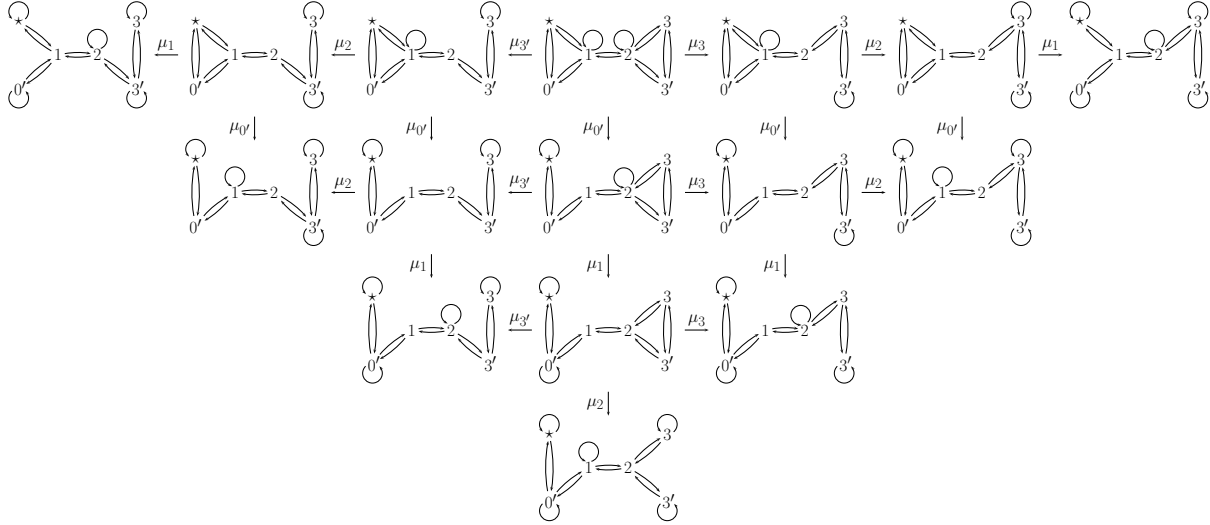
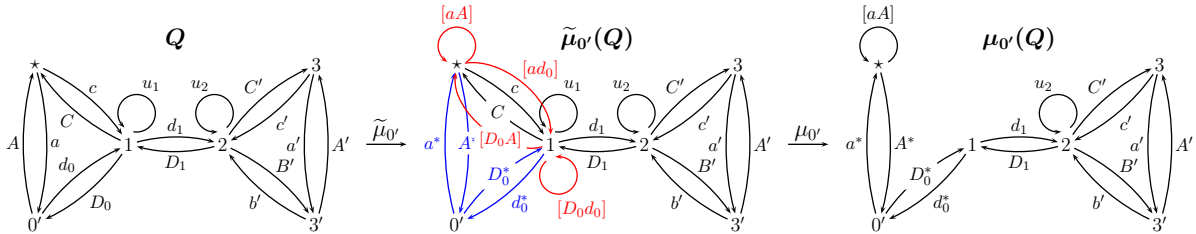


FIGURE 1. Mutations of type D_{12} .

We demonstrate how to calculate mutations of the QP (Q, W) by mutating at the vertex $0'$. First add new arrows $[aA], [D_0A], [ad_0], [D_0d_0]$, replace a, A, d_0, D_0 by a^*, A^*, d_0^*, D_0^* respectively as shown below, and denote the new quiver by $\tilde{\mu}_{0'}(Q)$.



Then the potential $[W] + \Delta$ is given by

$$\begin{aligned}
 [W] + \Delta = & -[ad_0]C - c[D_0A] + u_1Cc + u_1[D_0d_0] - u_1d_1D_1 + u_2d_1D_1 - u_2B'b' - u_2C'c' \\
 & - a'b'C' - A'c'b' + a^*[aA]A^* + a^*[ad_0]d_0^* + D_0^*[D_0A]A^* + D_0^*[D_0d_0]d_0^*,
 \end{aligned}$$

hence we have the mutation $\tilde{\mu}_{0'}(Q, W)$ of (Q, W) at the vertex $0'$. Moreover by taking derivations we have the following equalities

$$\begin{aligned} \partial_C &= -[ad_0] + cu_1, & \partial_c &= -[D_0A] + u_1C, & \partial_{u_1} &= [D_0d_0] - d_1D_1 + Cc, \\ \partial_{[ad_0]} &= -C + d_0^*a^*, & \partial_{[D_0A]} &= -c + A^*D_0^*, & \partial_{[D_0d_0]} &= u_1 + d_0^*D_0^*. \end{aligned}$$

Hence the reduced expression $W_{0'}$ of $[W] + \Delta$ is given by

$$W_{0'} = -(D_0^*d_0^*)^2 a^* A^* + d_0^* D_0^* d_1 D_1 + u_2 d_1 D_1 - u_2 B' b' - u_2 C' c' - a' b' C' - A' c' B' + a^* [aA] A^*.$$

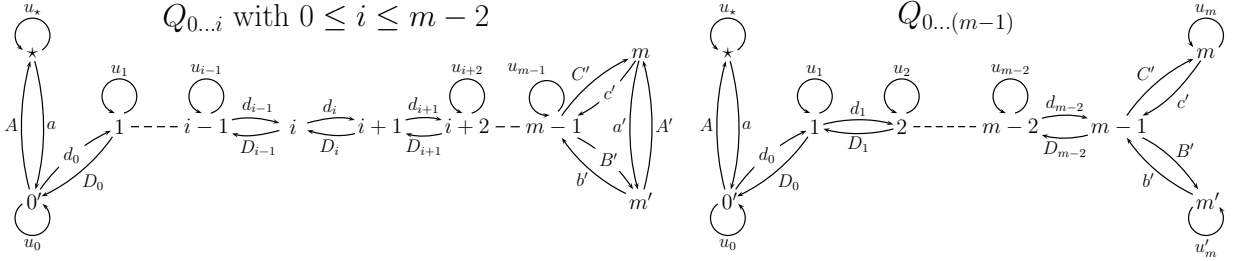
If $\mu_{0'}(Q)$ is a quiver obtained from $\tilde{\mu}_{0'}(Q)$ by removing $c, C, [ad_0], [D_0A], u_1, [D_0d_0]$, then $\mu_{0'}(Q, W) := (\mu_{0'}(Q), W_0)$ is a reduced QP and we have $\mathcal{P}(\tilde{\mu}_{0'}(Q, W)) \simeq \mathcal{P}(\mu_{0'}(Q, W))$. The others are obtained similarly.

Now we treat the general case. We write down all QPs which are obtained from the McKay QP by the rules of the game. The general case is similar to the case $n = 6$. Because of the symmetry between the vertices m and m' we only write down mutations of (Q, W) with respect to $0'$ and m . Mutations with respect to m' are done in the same way.

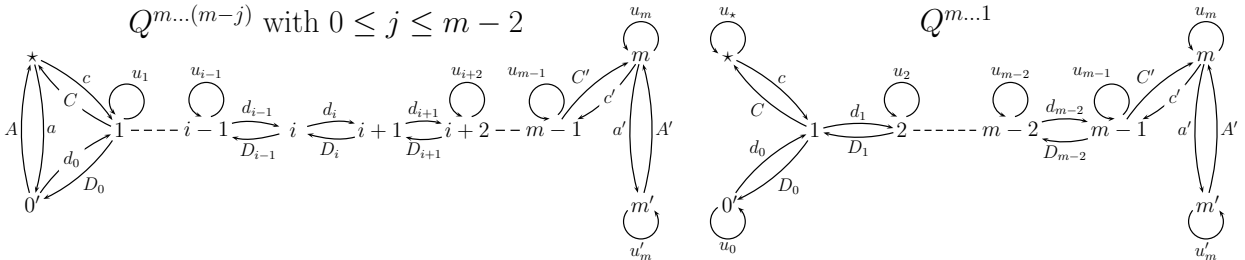
We first fix some notations. Note that we do not mutate at the vertex $\star (= 0)$ so Q_0, Q_{01}, \dots mean quivers obtained by mutating at $0'$. For simplicity, we do not use the notations $(-)^*$ and $[-]$: for example, starting from the McKay quiver Q with notations as in Section 3.2, arrows a, A and u_\star in the quiver $Q_{0\dots i}$ actually mean A^*, a^* and $[aA]$ respectively. Moreover we put

$$\begin{aligned} X_j &= u_j d_i D_j \text{ for } j = 0, \dots, m-2, & X_{m-1} &= u_{m-1} C' c', & X'_{m-1} &= u_{m-1} B' b' \\ Y_j &= u_j D_{j-1} d_{j-1} \text{ for } j = 1, \dots, m-1, & Y_m &= u_m c' C', & Y'_m &= u_m b' B' \\ Z_j &= d_j D_j D_{j-1} d_{j-1} \text{ for } j = 1, \dots, m-2, & Z_{m-1} &= C' c' D_{m-2} d_{m-2}, & Z'_{m-1} &= B' b' D_{m-2} d_{m-2}. \end{aligned}$$

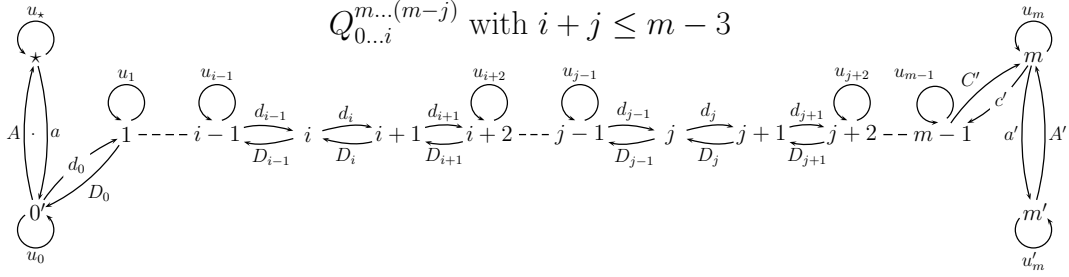
The list of mutated quivers $Q_{0\dots i}^{m\dots(m-j)}$ and their corresponding potentials $W_{0\dots i}^{m\dots(m-j)}$ is:



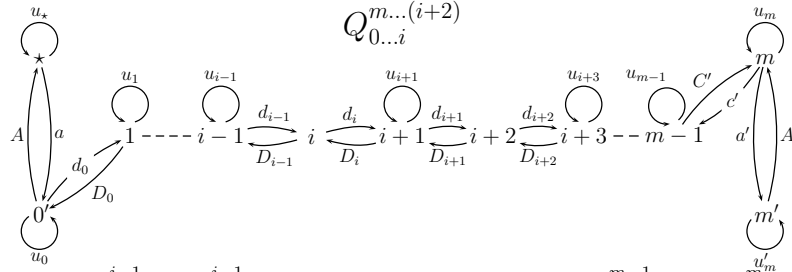
$$\begin{aligned} W_{0\dots i} &= aAu_\star - Aau_0^2 + \sum_{j=0}^{i-1} X_j - \sum_{j=1}^{i-1} Y_j - Z_i + Z_{i+1} + \sum_{j=i+2}^{m-2} X_j - \sum_{j=i+2}^{m-1} Y_j \\ &\quad + X_{m-1} + X'_{m-1} + a'b'C' + c'B'A', \\ W_{0\dots(m-1)} &= aAu_\star - Aau_0^2 + \sum_{j=0}^{m-3} X_j - \sum_{j=1}^{m-2} Y_j + Z_m + Z'_m + Y_m + Y'_m. \end{aligned}$$



$$\begin{aligned} W^{m\dots(m-j)} &= ad_0C + cD_0A - u_1D_0d_0 - u_1Cc + \sum_{j=1}^{i-1} X_j + \sum_{j=2}^{i-1} Y_j + Z_i - Z_{i+1} - \sum_{j=i+2}^{m-1} X_j + \sum_{j=i+2}^m Y_j \\ &\quad - u_m^2 A' a' + A' a' u'_m, \\ W^{m\dots 1} &= u_\star cC + u_0 d_0 D_0 - Ccd_1D_1 - D_0d_0d_1D_1 - \sum_{j=2}^{m-1} X_j + \sum_{j=3}^m Y_j - A' a' u_m^2 + a' A' u'_m. \end{aligned}$$



$$W_{0\dots i}^{m\dots(m-j)} = aAu_* - Aau_0^2 + \sum_{k=0}^{i-1} X_k - \sum_{k=1}^{i-1} Y_k - Z_i + Z_{i+1} + \sum_{k=i+2}^{j-1} X_k - \sum_{k=i+2}^{j-1} Y_k + Z_j - Z_{j+1} - \sum_{k=j+2}^{m-1} X_k + \sum_{k=j+2}^{m-1} Y_k - a'A'u_m^2 + A'a'u'_m.$$



$$W_{0\dots i}^{m\dots(i+2)} = aAu_* - Aau_0^2 + \sum_{k=0}^{i-1} X_k - \sum_{k=1}^{i-1} Y_k - Z_i + Y_{i+1} + X_{i+1} - Z_{i+2} - \sum_{k=i+3}^{m-1} X_k + \sum_{k=i+3}^{m-1} Y_k - a'A'u_m^2 + A'a'u'_m$$

The final remark shows that we have non-trivial mutations only if we follow the rule of the game (2) and it can be checked by direct calculations.

Lemma 5.3. For any (Q, W) as above, let $\Lambda = \mathcal{P}(Q, W)$ be the Jacobian algebra. Then for any vertex $i \in Q_0$ with a loop, $\dim_{\mathbb{C}} \Lambda_i = \infty$ holds.

5.3. The dihedral group of order $2n$ (n odd). Let $n = 2m + 1$ with $m \geq 2$ and (Q, W) be the McKay QP of G obtained in 3.3. In this case there are $m + 1$ non-equivalent mutation QPs which are equivalent to (Q, W) .

Example 5.4. Before treating the general case, we observe the case $n = 7$. In this case there are 4 non-equivalent QPs, see the Figure 2.

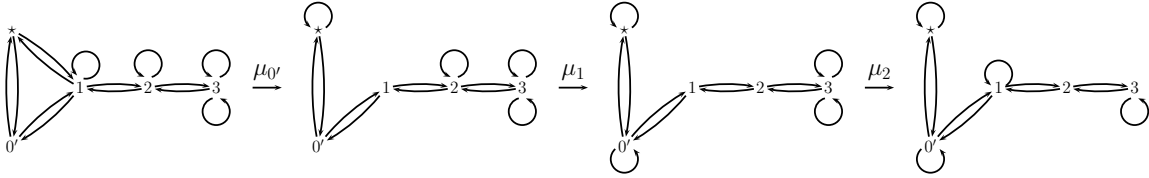
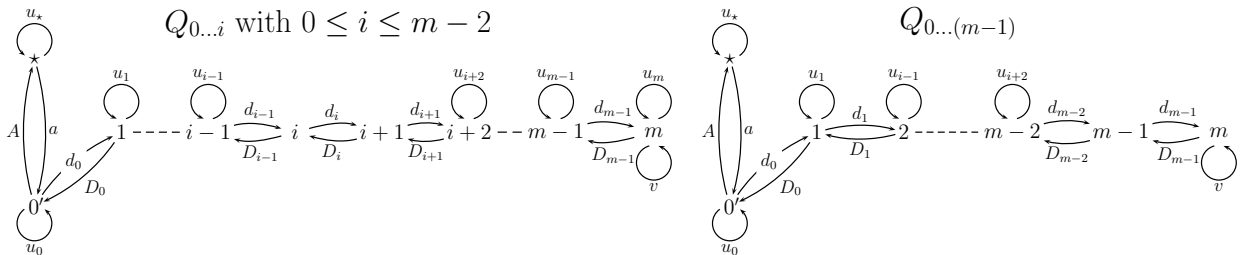


FIGURE 2. Mutations of type D_{14} .

Now we consider the general case. We write down all QPs which are obtained from the McKay QP by the rules of the game.



$$\begin{aligned}
W_{0\dots i} &= u_* aA - aAu_0^2 + \sum_{j=0}^{i-1} d_j D_j u_j - \sum_{j=1}^{i-1} D_{j-1} d_{j-1} u_j - D_{i-1} d_{i-1} D_i + D_i d_i d_{i+1} D_{i+1} \\
&\quad + \sum_{j=i+2}^{m-1} d_j D_j u_j - \sum_{j=i+2}^m D_{j-1} d_{j-1} u_j + u_m v^2. \\
W_{0\dots(m-1)} &= aAu_* - aAu_0^2 + \sum_{j=0}^{m-2} d_j D_j u_j - \sum_{j=1}^{m-2} D_{j-1} d_{j-1} u_j - D_{m-2} d_{m-2} d_{m-1} D_{m-1} + D_{m-1} d_{m-1} v^2.
\end{aligned}$$

The final remark shows that we have non-trivial mutations only if we follow the rule of the game (2) and it can be checked by direct calculations.

Lemma 5.5. For any (Q, W) as above, let $\Lambda = \mathcal{P}(Q, W)$ be the Jacobian algebra. Then for any vertex $i \in Q_0$ with a loop, $\dim_{\mathbb{C}} \Lambda_i = \infty$ holds.

5.4. The tetrahedral group. Let (Q, W) be the McKay QP of G obtained in 3.4. In this case there are 5 non-equivalent mutation QPs which are equivalent to (Q, W) . We list all of them in Figure 3.

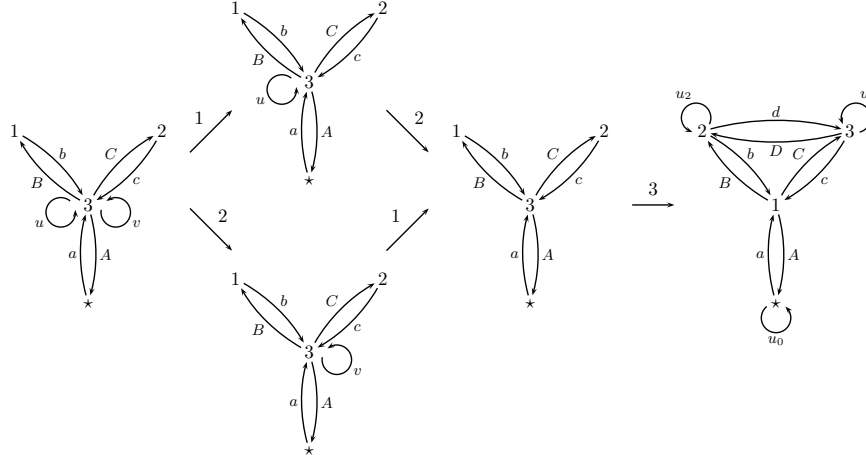


FIGURE 3. Mutations of type \mathbb{T} .

$$\begin{aligned}
W &= uAa + \omega uBb + \omega^2 uCc - \frac{1}{3}u^3 - vAa - \omega^2 vBb - \omega vCc + \frac{1}{3}v^3. \\
W_1 &= (1 - \omega^2)Aau - \omega AaBb + (\omega^2 - 1)Ccu - \omega^2 CcbB + \omega^2 Bbu^2 + \omega(Bb)^2u + \frac{1}{3}(Bb)^3. \\
W_2 &= (\omega^2 - 1)Aav - \omega AaCc + (1 - \omega^2)Bbv - \omega^2 BbCc + \omega^2 Ccv^2 + \omega(Cc)^2v + \frac{1}{3}(Cc)^3. \\
W_{12} &= AaBb + AaCc - (Bb)^2Cc - Bb(Cc)^2. \\
W_{123} &= aAu_0 - AaBb - AaCc + bBu_2 - dDu_2 + cCu_3 - Ddu_3 + Bdc + CDb.
\end{aligned}$$

We note that in (Q_1, W_1) and (Q_2, W_2) there is a relation of the form $v = \omega^2 u - \omega Bb$ and $u = \omega^2 v - \omega Cc$ respectively, so u and v are symmetric. We can easily check the following.

Lemma 5.6. For any (Q, W) as above, let $\Lambda = \mathcal{P}(Q, W)$ be the Jacobian algebra. Then for any vertex $i \in Q_0$ with a loop, $\dim_{\mathbb{C}} \Lambda_i = \infty$ holds.

5.5. The number of mutations of McKay QPs. We take a completion \widehat{R} of R and we want to look for all NCCRs over \widehat{R} . By the definition a NCCR over \widehat{R} is a nonsingular \widehat{R} -order $\Lambda = \text{End}_{\widehat{R}}(M)$ with a reflexive module M over \widehat{R} (see [IW] in detail). We say Λ is a *CM NCCR* if M is CM, equivalently M is a CT module. Thanks to Osamu Iyama, we know that the following is true even in the case of R not an isolated singularity: If there are only finite many number of non-isomorphic CT modules over \widehat{R} , then these are all CT modules. Since there are one-to-one correspondences between QPs obtained by a sequence of mutations from the McKay QP by our rule, CM NCCRs and CT modules over \widehat{R} , we have the next result.

Theorem 5.7. For a finite subgroup $G \subset SO(3)$ of type $\mathbb{Z}/n\mathbb{Z}$, D_{2n} and \mathbb{T} , the number of mutations of the McKay QP is finite up to isomorphisms. Moreover the number of CM NCCRs is finite up to Morita equivalences.

6. EXPLICIT DESCRIPTION OF THE MODULI SPACES \mathcal{M}_θ

In this section we describe the explicit structure of the crepant resolutions $\pi : X \rightarrow \mathbb{C}^3/G$ for $G \subset SO(3)$ of types $\mathbb{Z}/n\mathbb{Z}$, D_{2n} and \mathbb{T} . We do not describe the cyclic case $G \cong \mathbb{Z}/n\mathbb{Z}$ since $G\text{-Hilb}(\mathbb{C}^3)$ is the unique crepant resolution and it was already treated in [CR] and [Nak].

The results are summarized in Theorems 6.1, 6.3 and 6.5 for the subgroups of type D_{2n} with n odd, D_{2n} with n even and \mathbb{T} respectively. This explicit description allows us to conclude that every crepant resolution X is isomorphic to a moduli space \mathcal{M}_θ of θ -stable representations of the McKay quiver Q for some $\theta \in \Theta$, and X consists of a finite union of copies of \mathbb{C}^3 . The representation space of every open set can be found in the Appendix. Moreover, we give the local coordinates of every open set and the degrees of the normal bundles of the fibre $\pi^{-1}(0)$.

The proof is done by the explicit calculation of every case. The strategy used is the following:

- (1) We obtain first the crepant resolution $X := G\text{-Hilb}(\mathbb{C}^3) \cong \mathcal{M}_\theta$, where θ is the so called *0-generated stability condition* (i.e. θ is generic and $\theta_i > 0$ for every $i \neq 0$), which form [IN] is known to be contained in the chamber corresponding to $G\text{-Hilb}(\mathbb{C}^3)$. In particular, this choice of stability implies that there exist $\dim(\rho)$ linearly independent paths from ρ_0 to ρ .
- (2) By calculating the gluings between the open sets in X we obtain the degrees of the normal bundle of every rational exceptional curve in $\pi^{-1}(0)$. This is done by using the quiver of $\text{End}_{S^G}(\bigoplus_{\rho \in \text{Irr } G} S_\rho)$ where S_ρ are the CM S^G -modules $S_\rho := (S \otimes \rho)^G$, which coincides with the McKay quiver. By Lemma 8.1 we know that only the rational curves $E \subset X$ of type $(-1, -1)$ are floppable which gives us a finite number of possible flops of X .
- (3) We calculate the flop X' of X by only modifying the open sets containing the curve which is flopped, obtaining X' again as a moduli space $\mathcal{M}_{C'}$. Here we use a representation space which dominates both sides of the flop.
- (4) We repeat the process until we find all possible flops of $(-1, -1)$ -curves.

In this section (Q, R) always denote the McKay quiver with the relations R as in Section 3, $\Lambda := \mathbb{C}Q/R$ and $\mathbf{d} = (\dim \rho)_{\rho \in \text{Irr } G}$. Thus we write simply \mathcal{M}_θ or \mathcal{M}_C to denote the moduli space $\mathcal{M}_{\theta, \mathbf{d}}(\Lambda)$.

6.1. The dihedral group of order $2n$ (n odd). With the potential given in Section 3.3 the relations R in this case are:

$$\begin{aligned}
\partial a, \partial A, \partial b, \partial B, \partial c, \partial C : \quad & bC = 0, cB = 0, Ca = u_1B, Ac = bu_1, BA = u_1C, ab = cu_1. \\
\partial d_1, \dots, \partial d_{m-1} : \quad & D_1u_1 = u_2D_1, \dots, D_{m-1}u_{m-1} = u_mD_{m-1}. \\
\partial D_1, \dots, \partial D_{m-1} : \quad & u_1d_1 = d_1u_2, \dots, u_{m-1}d_{m-1} = d_{m-1}u_m. \\
\partial u_1 : \quad & Bb + Cc = d_1D_1. \\
\partial u_2, \dots, \partial u_{m-1} : \quad & d_2D_2 = D_1d_1, \dots, d_{m-1}D_{m-1} = D_{m-2}d_{m-2}. \\
\partial u_m : \quad & D_{m-1}d_{m-1} = v^2. \\
\partial v : \quad & u_mv + vu_m = 0.
\end{aligned}$$

We introduce the following notation for the arrows of Q as linear maps between vector spaces: $a := a$, $A := A$, $b := (b_1, b_2)$, $B := \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, $c := (c_1, c_2)$, $C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$, $d_i := \begin{pmatrix} d_{11}^i & d_{12}^i \\ d_{21}^i & d_{22}^i \end{pmatrix}$, $D_i := \begin{pmatrix} D_{11}^i & D_{12}^i \\ D_{21}^i & D_{22}^i \end{pmatrix}$, $u_j := \begin{pmatrix} u_{11}^j & u_{12}^j \\ u_{21}^j & u_{22}^j \end{pmatrix}$ and $v := \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$, where $1 \leq i \leq m-1$, $1 \leq j \leq m$ and every entry belongs to \mathbb{C} . We may drop the upper indices in the entries of the matrices for d_i , D_i and u_i if the context allows us to do so, and we may also drop the lower indices when there exist a unique element in the matrix which is non-constant.

Theorem 6.1. Let $G = D_{2n}$ with n odd. Then,

- (1) There are $m + 1$ crepant resolutions of \mathbb{C}^3/G given by the following open covers:

$$X_{0\dots i} = \bigcup_{k=1}^{i+1} U''_k \cup U'_{i+2} \cup \bigcup_{k=i+3}^{m+2} U_i, \text{ for } -1 \leq i \leq m,$$

where $U_i, U'_i, U''_i \cong \mathbb{C}^3$ for all i (see Appendix). For $i = -1$ we have $X = G\text{-Hilb}(\mathbb{C}^3)$.

- (2) Let $f_1 := x^{2m+1} + y^{2m+1}$ and $f_2 := x^{2m+1} - y^{2m+1}$. The local coordinates of the open sets are

$$\begin{aligned} U'_1 &\cong \mathbb{C}_{a,b,C}^3 = \text{Spec } \mathbb{C}[\frac{-z}{f_2}, 2xy, f_1]. \\ U_i &\cong \mathbb{C}_{d,D,u}^3 = \text{Spec } \mathbb{C}[\frac{-2x^{i-1}y^{i-1}z}{f_2}, \frac{-f_2}{x^{i-2}y^{i-2}z}, \frac{zf_1}{f_2}], \text{ for } i \leq m+1. \\ U_{m+2} &\cong \mathbb{C}_{v,V,u}^3 = \text{Spec } \mathbb{C}[z^2, \frac{f_1}{2x^m y^m}, \frac{-f_2}{2x^m y^m z}]. \\ U'_i &\cong \mathbb{C}_{a,d,D}^3 = \text{Spec } \mathbb{C}[\frac{-2x^{i-1}y^{i-1}z}{f_2}, \frac{-f_1}{2x^{i-1}y^{i-1}}, 2xy], \text{ for } i \leq m+1. \\ U''_i &\cong \mathbb{C}_{a,d,D}^3 = \text{Spec } \mathbb{C}[\frac{zf_1}{f_2}, \frac{-2x^i y^i}{f_1}, \frac{-f_1}{x^{i-1}y^{i-1}}], \text{ for } i \leq m. \end{aligned}$$

- (3) The degrees of the normal bundles $\mathcal{N}_{X/E}$ of the exceptional rational curves $E \subset X_{0\dots i}$ are

Open cover of E	Degree of $\mathcal{N}_{X/E}$
$U_i \cup U_{i+1}$	$(-2, 0)$ for $i = 2, \dots, m$ $(-3, 1)$ for $i = m+1$
$U'_i \cup U_{i+1}$	$(-1, -1)$ for $i = 1, \dots, m$ $(-2, 0)$ for $i = m+1$
$U''_i \cup U'_{i+1}$	$(-1, -1)$ for $i = 1, \dots, m-1$
$U''_i \cup U''_{i+1}$	$(-2, 0)$ for $i = 1, \dots, m$

- (4) The dual graph of the fibre over the origin for every crepant resolution is of the form:



Proof. Let $X := G\text{-Hilb}(\mathbb{C}^3) \cong \mathcal{M}_\theta$ for the 0-generated stability condition θ and dimension vector $\frac{1}{2} \dots 2$. Then we can construct the following open cover of X :

Claim 6.2. $G\text{-Hilb}(\mathbb{C}^3) \cong U'_1 \cup \bigcup_{i=2}^{m+2} U_i$.

Proof. We divide the proof into 4 steps:

Step1: We can always choose $c = (1, 0)$. Otherwise, by the 0-generated stability we need to have $ab = (1, 0)$, so that the relation $ab = cu_1$ implies that $A(c_1)^2 + u_{21}^1 c_2 = 1$. But then c_1 and c_2 cannot be both zero at the same time, which means that we can change basis at the vertex 1 to obtain $c = (1, 0)$.

Step2: Similarly, we can change basis to choose $d_i := \begin{pmatrix} 1 & 0 \\ d_{21}^i & d_{22}^i \end{pmatrix}$ for every i . Therefore it is remaining to generate the basis element $(0, 1)$ at every 2-dimensional vertex.

Step3: The open conditions to generate the 2-dimensional vertices can be done involving only the maps d_i and D_i . Indeed, if we suppose that $u_i = \begin{pmatrix} 0 & 1 \\ u_{21}^i & u_{22}^i \end{pmatrix}$ then using the relations involving the vertex i we can conclude that $D_{22}^i \neq 0$, thus we can choose $D_{21}^i = 0$ and $D_{22}^i = 1$ instead.

Step4: By the stability condition we need to reach the vertex $0'$ with a nonzero map from 0, thus we have that

$$\begin{aligned} &\text{either } cd_1 d_2 \cdots d_i D_i D_{i-1} \cdots D_1 B \neq 0 \\ &\text{or } a \neq 0 \end{aligned}$$

Consider the first case and suppose initially that $i = 1$, which after a change of basis is equivalent to say that $cd_1 D_1 B = 1$. In particular we can choose $B_2 = 1$ and $D_1 := \begin{pmatrix} 0 & 1 \\ D_{21}^1 & D_{22}^1 \end{pmatrix}$, which leads to a contradiction when applying the relation $Bb + Cc = d_1 D_1$. Similarly, for

$i > 0$ we obtain contradiction with the relation $D_{i-1}d_{i-1} = d_i D_i$. Therefore we either have $cd_1 \cdots d_{m-1}u_m D_{m-1} \cdots D_1 B \neq 0$ or $cd_1 \cdots d_{m-1}v D_{m-1} \cdots D_1 B \neq 0$. By the Step 2 we can always choose the second option, which gives the open set

$$U'_1 : cd_1 \cdots d_{m-1}v D_{m-1} \cdots D_1 B = 1$$

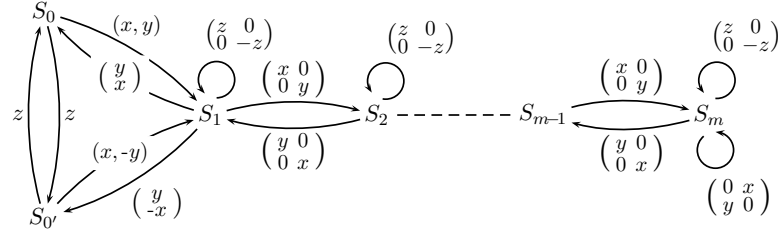
By the relation $cB = 0$ we get $B_1 = 0$, so by changing basis we can always take $c = (1, 0)$, $d_i := \begin{pmatrix} 1 & 0 \\ d_{21}^i & d_{22}^i \end{pmatrix}$, $D_i := \begin{pmatrix} d_{11}^i & d_{12}^i \\ 0 & 1 \end{pmatrix}$ for all i , $v = \begin{pmatrix} 0 & 1 \\ v_{21} & v_{22} \end{pmatrix}$ and $B := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and using the relations we obtain the representation space for U'_1 shown in the Appendix.

If we suppose that $a \neq 0$, by Step 2 and the usual change of basis at every 2-dimensional vertex i we reach the standard basis element $(1, 0)$ by the path $cd_1 \cdots d_i$. Then, the rest of possibilities for the open sets are

$$\begin{aligned} U_2 : & \quad cd_1 \cdots d_{m-1}v D_{m-1} \cdots D_1 = (0, 1), a = 1 \\ U_3 : & \quad cd_1 \cdots d_{m-1}v D_{m-1} \cdots D_2 = (0, 1), ab = 1 \\ U_i : & \quad cd_1 \cdots d_{m-1}v D_{m-1} \cdots D_i = (0, 1), abd_1 \cdots d_{i-1} = (0, 1) \text{ for } 4 \leq i < m \\ U_{m+1} : & \quad cd_1 \cdots d_{m-1}v = (0, 1), abd_1 \cdots d_{m-2} = (0, 1) \\ U_{m+2} : & \quad cd_1 \cdots d_{m-1} = (1, 0), abd_1 \cdots d_{m-1} = (0, 1) \end{aligned}$$

Using the relations in every case we obtain the representation spaces given in the Appendix, and we can conclude that $G\text{-Hilb}(\mathbb{C}^3)$ is covered by the union of $m + 2$ open sets isomorphic each of them to \mathbb{C}^3 . \square

Let $S_\rho := (S \otimes \rho)^G$ be the CM S^G -modules. Writing down the irreducible maps between these modules we obtain the following McKay quiver:



Now let $U_{\alpha, \beta, \gamma} \subset X_{0 \dots i}$ an open set and consider the corresponding representation space as shown in the Appendix. Every point in $U_{\alpha, \beta, \gamma}$ is a representation of Q of dimension vector $d = \frac{1}{2} 1 \dots 2$ generated by a subset of linearly independent *distinguished arrows*. In particular, if $X \cong G\text{-Hilb}(\mathbb{C}^3)$ then $U_{\alpha, \beta, \gamma}$ is generated from ρ_0 , otherwise generated from ρ_0 and $\rho_{0'}$. We choose the basis elements of the vector spaces at every vertex of Q by following the distinguished arrows of $U_{\alpha, \beta, \gamma}$ in the quiver between the CM R -modules shown above. For instance, in U'_1 with coordinates a, b and C , the distinguished arrows are $c, d_1, \dots, d_{m-1}, v, D_{m-1}, \dots, D_1$ and B , so we choose as the basis elements:

V_0	1	V_2	$(x^2, y^2) = e_1$
$V_{0'}$	$-x^7 + y^7$	V_3	$(y^5, x^5) = e_2$
V_1	$(x, y) = e_1$		$(x^3, y^3) = e_1$
	$(y^6, x^6) = e_2$		$(y^4, x^4) = e_2$

In particular, this choice implies that $z = a(-x^7 + y^7)$, $x^7 + y^7 = -b \cdot 1$ and $2xy = C \cdot 1$, which by rescaling the coefficients gives us the coordinates of the open set U'_1 : $a = z/(x^7 - y^7)$, $b = (x^7 + y^7)$ and $C = 2xy$. Knowing the representation space of an open set $U \subset \mathcal{M}_\theta$, this method provides local coordinates for U .

It follows that in X we have $\pi^{-1}(0) = \bigcup_{i=0}^m E_i$ where $E_i \cong \mathbb{P}^1$ with coordinates $(xy)^i z : x^{2m+1} - y^{2m+1}$ for $i = 0, \dots, m$, intersecting pairwise according to the dual graph shown below (see also [GNS2] §3.5).

$$\begin{array}{ccccccc} E_0 & E_1 & & E_{m-1} & E_m \\ \bullet & \bullet & \cdots & \bullet & \bullet \end{array}$$

Therefore E_0 is a $(-1, -1)$ -curve, E_m is a $(-3, 1)$ -curve, and the rest are $(-2, 0)$ -curves. By Lemma 8.1 only E_0 can be floppable, which gives us X_0 .

Recall that any $(-1, -1)$ -curve $E \subset X$ is floppable. In fact, there exists $F \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset Y$ fitting in the following diagram

$$\begin{array}{ccc} & F \subset Y & \\ \sigma \swarrow & & \searrow \sigma' \\ E \subset X & \dashrightarrow & X' \supset E' \end{array}$$

where F dominates E and E' in two different ways as the exceptional locus for σ and σ' (see for instance [Pagoda] 5.5 and references therein). In our case, the curve E_0 is covered by U'_1 and U_2 , and to calculate the new open sets in $U''_1, U'_2 \subset X_0$ covering the flopped curve E'_0 we look at the following representation space:

$$\begin{array}{c} \begin{array}{ccccccc} 0 & & (1, 0) & \begin{pmatrix} ab_1 & ab_2 \\ -ab_2(b_2B)^{2m-1} & -ab_1 \end{pmatrix} & \begin{pmatrix} ab_1 & ab_2^2B \\ -ab_2(b_2B)^{2m-2} & -ab_1 \end{pmatrix} & \begin{pmatrix} ab_1 & ab_2(b_2B)^{m-1} \\ -ab_2(b_2B)^m & -ab_1 \end{pmatrix} \\ & \searrow & \begin{pmatrix} b_2B \\ -b_1B \end{pmatrix} & \curvearrowright & \begin{pmatrix} 1 & 0 \\ 0 & b_2B \end{pmatrix} & \curvearrowright & \begin{pmatrix} 1 & 0 \\ 0 & b_2B \end{pmatrix} \\ ab_1^2 - ab_2(b_2B)^{2m-1} & \xrightarrow{a} & 1 & \xleftrightarrow{\begin{pmatrix} b_2B & 0 \\ 0 & 1 \end{pmatrix}} & 2 & \cdots & m-1 & \xleftrightarrow{\begin{pmatrix} b_2B & 0 \\ 0 & 1 \end{pmatrix}} & m \\ & \searrow & \begin{pmatrix} b_1 & b_2 \end{pmatrix} & \curvearrowright & \begin{pmatrix} 0 & 1 \\ b_2B & 0 \end{pmatrix} & \curvearrowright & \begin{pmatrix} 0 & 1 \\ b_2B & 0 \end{pmatrix} \\ & \searrow & \begin{pmatrix} 0 \\ B \end{pmatrix} & & & & & & \end{array} \end{array}$$

It is the representation space obtained by taking the distinguished nonzero arrows which are common in U'_1 and U_2 . This space dominates both sides of the flop, in the sense that if we set $B = 1$ we get U'_1 and if we set $a = 1$ we get U_2 , covering E_0 ; if $b_1 = 1$ we get U''_1 and if $b_2 = 1$ we get U'_2 , covering the flop of E_0 . The coordinates of $F \cong \mathbb{P}^1 \times \mathbb{P}^1$ are $(a : B; b_1 : b_2)$.

For the rest of crepant resolutions $X_{0\dots i}$ we argue in the same way, i.e. we repeat this process in every $(-1, -1)$ -curve to produce every open set $U \subset \mathcal{M}_\theta$ shown in the Appendix. For $0 < i < m$ notice that in each of them there are precisely two $(-1, -1)$ -curves, whose flops give $X_{0\dots(i-1)}$ and $X_{0\dots(i+1)}$ respectively. In the case $i = m$, the 3-fold $X_{0\dots m}$ has only E_{m-1} as a $(-1, -1)$ -curve and the flop produces the previous $X_{0\dots m-1}$, so we stop.

The description of the affine coordinates in (2) gives the gluings between the open sets shown below. From them we can read off the degrees of the normal bundles, and (3) follows.

$$\begin{aligned} U'_1 \rightarrow U_2 : & (a, b, C) \mapsto (ab, a^{-1}, -aC) \\ U_i \rightarrow U_{i+1} : & (d, D, u) \mapsto (\tfrac{1}{2}d^2D, 2d^{-1}u), \text{ for } i = 2, \dots, m+1 \\ U_{m+1} \rightarrow U_{m+2} : & (d, D, u) \mapsto (u^2 - \tfrac{1}{2}d^3D, -d^{-1}u, d^{-1}) \\ U'_2 \rightarrow U_3 : & (a, b, B) \mapsto (\tfrac{1}{2}aB, 2b^{-1}, ab) \\ U'_i \rightarrow U_{i+1} : & (a, d, D) \mapsto (\tfrac{1}{2}aD, 2a^{-1}, ad), \text{ for } i = 3, \dots, m \\ U'_{m+1} \rightarrow U_{m+2} : & (a, d, D) \mapsto (a^2d^2 - \tfrac{1}{4}a^2D^2, -d, a^{-1}) \\ U''_1 \rightarrow U'_2 : & (a, b, B) \mapsto (ab, b^{-1}, bB) \\ U''_i \rightarrow U'_{i+1} : & (a, d, D) \mapsto (ad, d^{-1}, dD), \text{ for } i = 2, \dots, m-1 \\ U''_i \rightarrow U''_{i+1} : & (a, d, D) \mapsto (a, \tfrac{1}{2}d^2D, 2d^{-1}), \text{ for } i = 1, \dots, m \end{aligned}$$

In every crepant resolution the dual graph of the fibre over the origin $0 \in \mathbb{C}^3/G$ is the same as $X = G\text{-Hilb}(\mathbb{C}^3)$, they only differ in the degrees of the normal bundles over E_i given in (2). \square

6.2. The dihedral group of order $2n$ (n even). With the potential given in Section 3.2 the list of relations in this case is the following:

$$\begin{aligned}
\partial a, \partial A, \partial b, \partial B, \partial c, \partial C : & \quad bC = 0, cB = 0, Ca = u_1B, Ac = bu_1, BA = u_1C, ab = cu_1. \\
\partial a', \partial A', \partial b', \partial B', \partial c', \partial C' : & \quad b'C' = 0, c'B' = 0, C'a' = u_{m-1}B', \\
& \quad A'c' = b'u_{m-1}, B'A' = u_{m-1}C', a'b' = c'u_{m-1}. \\
\partial d_1, \dots, \partial d_{m-2} : & \quad D_1u_1 = u_2D_1, \dots, D_{m-2}u_{m-2} = u_{m-1}D_{m-2}. \\
\partial D_1, \dots, \partial D_{m-2} : & \quad u_1d_1 = d_1u_2, \dots, u_{m-2}d_{m-2} = d_{m-2}u_{m-1}. \\
\partial u_1 : & \quad Bb + Cc = d_1D_1. \\
\partial u_2, \dots, \partial u_{m-2} : & \quad d_2D_2 = D_1d_1, \dots, d_{m-2}D_{m-2} = D_{m-3}d_{m-3}. \\
\partial u_m : & \quad B'b' + C'c' = D_{m-2}d_{m-2}.
\end{aligned}$$

We consider the same notation as in 6.1 for the arrows of Q as linear maps, adding in this case $a' := a'$, $A' := A'$, $b' := (b'_1, b'_2)$, $B' := \begin{pmatrix} B'_1 \\ B'_2 \end{pmatrix}$, $c' := (c'_1, c'_2)$ and $C' := \begin{pmatrix} C'_1 \\ C'_2 \end{pmatrix}$.

Theorem 6.3. Let $G = D_{2n}$ with n even. Then,

- (1) There are $(m+1)(m+2)/2$ non-isomorphic crepant resolutions of \mathbb{C}^3/G given by the following open covers:

$$X_{0\dots i}^{m\dots(m-j)} = \bigcup_{k=1}^{i+1} U_k'' \cup U_{i+2}' \cup \bigcup_{k=i+3}^{m-j} U_k \cup V_{m-j+1}' \cup \bigcup_{k=m-j+2}^{m+3} V_k''$$

for $-1 \leq i, j \leq m-1$, where $U_k, U_k', U_k'', V_k', V_k'' \cong \mathbb{C}^3$ for all k (see Appendix). When $i = j = -1$ we have $X = G\text{-Hilb}(\mathbb{C}^3)$.

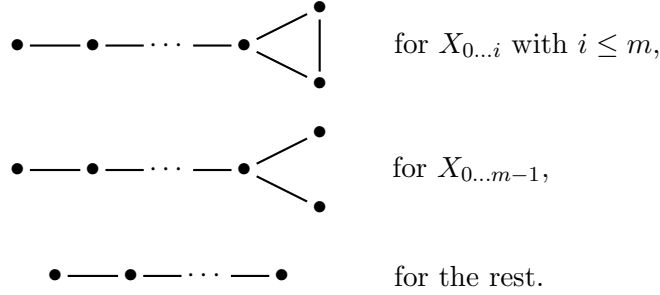
- (2) Let $f_1 := x^m + y^m$ and $f_2 := x^m - y^m$. The local coordinates of the open sets are

$$\begin{aligned}
U_i &\cong \mathbb{C}_{d,D,u}^3 = \text{Spec } \mathbb{C}[\frac{-2x^{i-1}y^{i-1}z}{f_1f_2}, \frac{-f_1f_2}{x^{i-2}y^{i-2}z}, \frac{zf_1}{f_2}], \text{ for } i \leq m+1 \\
U_{m+1} &\cong \mathbb{C}_{B',c',C}^3 = \text{Spec } \mathbb{C}[\frac{zf_1}{f_2}, \frac{-f_1f_2}{2x^{m-1}y^{m-1}z}, \frac{zf_2}{f_1}] \\
U_1' &\cong \mathbb{C}_{a,b,C}^3 = \text{Spec } \mathbb{C}[\frac{-z}{f_1f_2}, 2xy, f_1^2] \\
U_i' &\cong \mathbb{C}_{a,d,D}^3 = \text{Spec } \mathbb{C}[\frac{-2x^{i-1}y^{i-1}z}{f_1f_2}, \frac{-f_1^2}{2x^{i-1}y^{i-1}}, 2xy], \text{ for } 2 \leq i \leq m \\
U_{m+1}' &\cong \mathbb{C}_{a,B',c'}^3 = \text{Spec } \mathbb{C}[\frac{zf_2}{f_1}, \frac{f_1^2}{f_2^2}, \frac{-f_2^2}{2x^{m-1}y^{m-1}}] \\
U_i'' &\cong \mathbb{C}_{a,d,D}^3 = \text{Spec } \mathbb{C}[\frac{zf_1}{f_2}, \frac{-2x^iy^i}{f_1^2}, \frac{-f_1^2}{x^{i-1}y^{i-1}}], \text{ for } i \leq m-1 \\
U_m'' &\cong \mathbb{C}_{a,c',C'}^3 = \text{Spec } \mathbb{C}[\frac{zf_1}{f_2}, \frac{-f_1^2}{2x^{m-1}y^{m-1}}, \frac{f_2^2}{f_1^2}] \\
V_i' &\cong \mathbb{C}_{a',u,C'}^3 = \text{Spec } \mathbb{C}[\frac{-x^{i-2}y^{i-2}z^2}{f_2^2}, 2xy, \frac{-2^{m-i}f_1f_2}{x^{i-2}y^{i-2}z}], \text{ for } i \leq m \\
V_{m+1}' &\cong \mathbb{C}_{a',b',C'}^3 = \text{Spec } \mathbb{C}[\frac{-2x^{m-1}y^{m-1}z^2}{f_2^2}, \frac{-f_2^2}{2x^{m-1}y^{m-1}}, \frac{-f_1f_2}{2x^{m-1}y^{m-1}z}] \\
V_{m+2}' &\cong \mathbb{C}_{a',c',C'}^3 = \text{Spec } \mathbb{C}[z^2, \frac{-f_2^2}{2x^{m-1}y^{m-1}}, \frac{f_1}{zf_2}] \\
V_i'' &\cong \mathbb{C}_{d,D,C'}^3 = \text{Spec } \mathbb{C}[\frac{-2x^{i-2}y^{i-2}z^2}{f_2^2}, \frac{-f_2^2}{x^{i-3}y^{i-3}z^2}, \frac{zf_1}{f_2}], \text{ for } i \leq m+1 \\
V_{m+2}'' &\cong \mathbb{C}_{A,c',C'}^3 = \text{Spec } \mathbb{C}[z^2, \frac{-f_2^2}{2x^{m-1}y^{m-1}z^2}, \frac{zf_1}{f_2}] \\
V_{m+3}'' &\cong \mathbb{C}_{A',b',B'}^3 = \text{Spec } \mathbb{C}[z^2, \frac{f_1^2}{2x^{m-1}y^{m-1}}, \frac{f_2}{zf_1}]
\end{aligned}$$

- (3) The degrees of the normal bundles $\mathcal{N}_{X/E}$ of the exceptional rational curves $E \subset X_{0\dots i}^{m\dots(m-j)}$ are

Open cover of E	Degree of $\mathcal{N}_{X/E}$	Open cover of E	Degree of $\mathcal{N}_{X/E}$
$U_i \cup U_{i+1}$	$(-2, 0)$ for $2 \leq i \leq m$	$U'_i \cup V'_{i+1}$	$(-2, 0)$ for $1 \leq i \leq m$
$U_i \cup V'_{i+1}$	$(-1, -1)$ for $2 \leq i \leq m+1$	$U''_i \cup U'_{i+1}$	$(-1, -1)$ for $1 \leq i \leq m$
$U_{m+1} \cup V''_{m+3}$	$(-1, -1)$	$U''_i \cup U''_{i+1}$	$(-2, 0)$ for $1 \leq i \leq m-1$
$U'_i \cup U_{i+1}$	$(-1, -1)$ for $1 \leq i \leq m$	$V'_i \cup V''_{i+1}$	$(-1, -1)$ for $2 \leq i \leq m+1$
	$(-2, 0)$ for $i = m+1$	$V''_i \cup V'_{i+1}$	$(-2, 0)$ for $3 \leq i \leq m+2$

- (4) Let $\pi_{ij} : X_{0\dots i}^{m\dots(m-j)} \rightarrow \mathbb{C}^3/G$ a crepant resolution. The dual graph of $\pi_{ij}^{-1}(0)$ is:



Proof. As in the proof of Claim 6.2 we start by calculating explicitly $X := G\text{-Hilb}(\mathbb{C}^3) \cong \mathcal{M}_\theta$ for the 0-generated stability condition θ .

Claim 6.4. $G\text{-Hilb}(\mathbb{C}^3) \cong U'_1 \cup \bigcup_{k=2}^{m+1} U_k \cup V'_{m+2} \cup V''_{m+3}$.

Proof. From Section 3 we can see that the McKay quiver in this case only differs from the case when n is odd in the vertices $m-1$, m and m' , thus the argument is very similar to the proof of the Claim 6.2. In particular, we can choose $c = (1, 0)$, $d_i := \begin{pmatrix} 1 & 0 \\ d_{21}^i & d_{22}^i \end{pmatrix}$ and we cannot have a path $cd_i \cdots d_i D_i \cdots D_1 B \neq 0$ for any i . Therefore, we have three possibilities to reach the second linearly independent vector, which we may choose to be $(0, 1)$, at the vector space at the vertex $m+1$. Namely $cd_i \cdots d_{m-2} B' b' = (0, 1)$, $cd_i \cdots d_{m-2} C' c' = (0, 1)$ or $abd_i \cdots d_{m-2} = (0, 1)$. By symmetry the first two are equivalent, so we can assume that $C'_1 = 1$ and $c' = (0, 1)$. In other words, we have that

$$\begin{aligned} &\text{either } cd_1 d_2 \cdots d_{m-2} C' c' = (0, 1) \\ &\text{or } abd_i \cdots d_{m-2} = (0, 1) \end{aligned}$$

Let us consider the first case. To reach the 1-dimensional vector space at the vertex m' we can always choose $B'_1 = 1$. Indeed, by the relations $c' B' = 0$ and $C' a' = u_{m-1} B'$ we obtain the equality $a' = u_{11} B'_1$. This means that if we choose $a' \neq 0$ then $B'_1 \neq 0$, and we can change basis to consider $B'_1 = 1$ instead.

Thus, arguing as in the proof of Claim 6.2 we obtain the following open sets

$$\begin{aligned} U'_1 : & \quad cd_1 \cdots d_{m-2} C' = 1, B' = 1, c' D_{m-2} \cdots D_1 B = 1 \\ U_2 : & \quad cd_1 \cdots d_{m-2} C' = 1, B' = 1, c' D_{m-2} \cdots D_1 = (0, 1), a = 1 \\ U_3 : & \quad cd_1 \cdots d_{m-2} C' = 1, B' = 1, c' D_{m-2} \cdots D_2 = (0, 1), ab = (0, 1) \\ U_i : & \quad cd_1 \cdots d_{m-2} C' = 1, B' = 1, c' D_{m-2} \cdots D_{i-2} = (0, 1), abd_1 \cdots d_{i-3} = (0, 1) \text{ for } 4 \leq i < m+1 \end{aligned}$$

If $abd_i \cdots d_{m-2} = (0, 1)$ then there are only two possibilities which satisfy the 0-generated stability condition, namely $cd_1 \cdots d_{m-2} B' A' = 1$ and $cd_1 \cdots d_{m-2} C' a' = 1$, giving the open sets V'_{m+2} and V''_{m+3} :

$$\begin{aligned} V'_{m+2} : & \quad abd_i \cdots d_{m-2} = (0, 1), cd_1 \cdots d_{m-2} B' A' = 1 \\ V''_{m+3} : & \quad abd_i \cdots d_{m-2} = (0, 1), cd_1 \cdots d_{m-2} C' a' = 1 \end{aligned}$$

Again, by using the relations in every case we obtain the representation spaces given in the Appendix, and we can conclude that $G\text{-Hilb}(\mathbb{C}^3)$ is covered by the union of $m + 3$ open sets isomorphic to \mathbb{C}^3 . \square

As in the proof of 6.1, the local coordinates are obtained using the quiver structure of the CM S^G -modules S_ρ , which in this case is given in Figure 4.

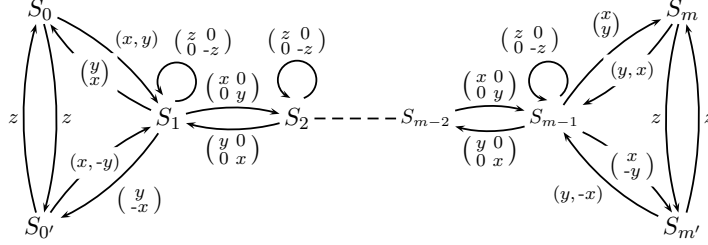
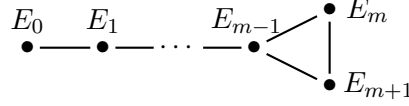


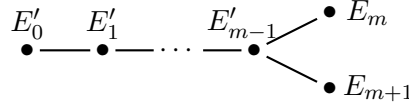
FIGURE 4. Quiver of the CM-modules S_ρ .

It follows that $\pi^{-1}(0) = \bigcup_{i=0}^{m+1} E_i$ where $E_i \cong \mathbb{P}^1$ intersect according to the following dual graph:



The curves E_i have coordinates $(xy)^i z : x^{2m} - y^{2m}$ for $i < m$, E_m has coordinates $x^m + y^m : z(x^m - y^m)$ and E_{m+1} has coordinates $x^m - y^m : z(x^m + y^m)$. The rational curves E_0 , E_m and E_{m+1} are $(-1, -1)$ -curves, and the rest of E_i 's are $(-2, 0)$ -curves. By Lemma 8.1 only the flop of E_0 , E_m and E_{m+1} gives us new crepant resolutions. As in Section 5.2, by the symmetry of the curves E_m and E_{m+1} it is enough to consider flops from $G\text{-Hilb}(\mathbb{C}^3)$ at E_0 and E_m .

Using the same method as in the proof of Claim 6.2 it can be checked that we can flop consecutively the curves E_0, \dots, E_{m-1} (in this order) to obtain the chain of flops $G\text{-Hilb}(\mathbb{C}^3) = X \dashrightarrow X_0 \dashrightarrow \dots \dashrightarrow X_{0\dots m-1}$. At every step we obtain the open sets shown in the Appendix and the local coordinates as in (2). The dual graph of these crepant resolutions are the same as the dual graph of $G\text{-Hilb}(\mathbb{C}^3)$ except for $X_{0\dots m-1}$ which is



In any of the crepant resolutions $X_{0\dots i}$ except for $i = m - 1$ we can flop the rational curve E_m to obtain the crepant resolution $X_{0\dots i}^m$, where now E_{m-1} and the flopped curve E'_m are $(-1, -1)$ -curves. Flopping E'_m again takes us back to $X_{0\dots i}$ and flopping E_{m-1} leads us to $X_{0\dots i}^{m(m-1)}$. In the same way we obtain the sequence of flops $X_{0\dots i} \dashrightarrow X_{0\dots i}^m \dashrightarrow X_{0\dots i}^{m(m-1)} \dashrightarrow \dots \dashrightarrow X_{0\dots i}^{m\dots i+2}$, and continuing the process in the same fashion we construct the $(m+1)(m+2)/2$ crepant resolutions $\pi : X_i^j \rightarrow \mathbb{C}^3/G$.

Except for $X_{0\dots i}$ which is described above, the dual graph of the fibre $\pi^{-1}(0)$ is



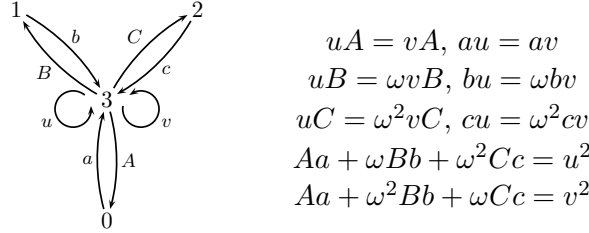
where the degrees of the normal bundles in each case change at each step according to (3). These degrees are obtained by using the gluings between the open sets shown below, and the

result follows.

$$\begin{array}{l}
 U_i \rightarrow U_{i+1} : (d, D, u) \mapsto (\frac{1}{2}d^2D, 2d^{-1}, u), i \leq m-1 \\
 U_m \rightarrow U_{m+1} : (d, D, u) \mapsto (u, d^{-1}, u - d^2D) \\
 U_{m+1} \rightarrow V'_{m+2} : (B', c', C') \mapsto (B'C', c'C', (C')^{-1}) \\
 U_{m+1} \rightarrow V'_{m+3} : (B', c', C') \mapsto (B'C', B'c', (B')^{-1}) \\
 U'_i \rightarrow U_{i+1} : (a, b, C) \mapsto (-aC, a^{-1}, ab), i = 1 \\
 \quad (a, d, D) \mapsto (\frac{1}{2}aD, 2a^{-1}, ad), 2 \leq i \leq m+1 \\
 \quad (a, d, D) \mapsto (ad, a^{-1}, ad + aD), i = m \\
 \quad (a, B', c') \mapsto (a^2B', c', a^{-1}), i = m+1 \\
 U'_i \rightarrow V'_{i+1} : (a, b, C) \mapsto (a^2C, b, 2^{m-2}a^{-1}), i = 1 \\
 \quad (a, d, D) \mapsto (\frac{1}{2}a^2d, D, 2^{m-i+1}a^{-1}), i \leq m-1 \\
 \quad (a, d, D) \mapsto (a^2d, d + D, a^{-1}), i = m
 \end{array}
 \quad \left| \quad
 \begin{array}{l}
 U''_i \rightarrow U'_{i+1} : (a, d, D) \mapsto (ad, d^{-1}, dD), i \leq m-1 \\
 \quad (a, c', C') \mapsto (aC', (C')^{-1}, c'C'), i = m \\
 U''_i \rightarrow U'_{i+1} : (a, d, D) \mapsto (a, \frac{1}{2}d^2D, 2d^{-1}), i \leq m-2 \\
 \quad (a, d, D) \mapsto (a, d^{-1}, 1 + d^2D), i = m-1 \\
 U_i \rightarrow V'_{i+1} : (d, D, u) \mapsto (\frac{1}{2}du, dD, 2^{m-i+1}d^{-1}), i \leq m-1 \\
 \quad (d, D, u) \mapsto (du, d^{-1}u + dD, d^{-1}), i = m \\
 V'_i \rightarrow V'_{i+1} : (a', u, C') \mapsto (a'u, (a')^{-1}, 2^{i-m}a'C'), i \leq m \\
 \quad (a', b', C') \mapsto (a'b', (a')^{-1}, a'C'), i = m+1 \\
 V''_i \rightarrow V'_{i+1} : (d, D, C') \mapsto (\frac{1}{2}d^2D, d^{-1}, C'), i \leq m \\
 \quad (d, D, C') \mapsto ((C')^2 + d^2D, d^{-1}, C'), i = m+1 \\
 \quad (A, c', C') \mapsto (A, c'(C')^2, (C')^{-1}), i = m+2
 \end{array}$$

□

6.3. The tetrahedral group. Let G be the tetrahedral group of order 12 and let $S := \mathbb{C}[x, y, z]$ as usual. In this case the McKay quiver and the relations R are the following:



Considering the arrows as linear maps between vector spaces we denote them by $a := (a_1, a_2, a_3)$, $A := \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$, $b := (b_1, b_2, b_3)$, $B := \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$, $c := (c_1, c_2, c_3)$, $C := \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$, $u := \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}$ and $v := \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}$. Let us define the following polynomials which appear frequently in the rest of this section.

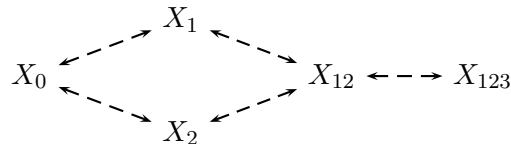
$$\begin{array}{ll}
 f_0 := x^2 + y^2 + z^2, & R_0 := y^2z^2 + x^2z^2 + x^2y^2, \\
 f_1 := x^2 + \omega^2y^2 + \omega z^2, & R_1 := y^2z^2 + \omega x^2z^2 + \omega^2x^2y^2, \\
 f_2 := x^2 + \omega y^2 + \omega^2z^2, & R_2 := y^2z^2 + \omega^2x^2z^2 + \omega x^2y^2. \\
 f_3 := xyz, & \\
 f_4 := (x^2 - y^2)(y^2 - z^2)(z^2 - x^2). &
 \end{array}$$

Notice that we have $3R_0 = f_0^2 - f_1f_2$, $3R_1 = f_1^2 - f_0f_2$, $3R_2 = f_2^2 - f_0f_1$, $R_0^3 = R_1R_2 + 3f_0f_3^2$, $R_1^3 = R_0R_2 + 3f_1f_3^2$, $R_2^3 = R_0R_1 + 3f_2f_3^2$ and $R_2^3 - R_1^3 = 3f_3^2(f_2R_2 - f_1R_1)$, as some of the relations among these polynomials. The invariant ring S^G is generated by f_0, f_3, f_1f_2 and f_4 (See [GNS1, §2]) but $f_1f_2 = f_0^2 - 3R_0$ holds, hence one can take R_0 as a generator of S^G instead of f_1f_2 . There is only one relation between these polynomials:

$$f_4^2 + 4R_0^3 - f_0^2R_0^2 - 18f_0R_0f_3^2 + 4f_0^3f_3^2 + 27f_3^4.$$

Theorem 6.5. Let G be the tetrahedral group of order 12 and let $\pi : Y \rightarrow \mathbb{C}^3/G$ be a crepant resolution. Then,

- (1) There exist 5 crepant resolutions of $\pi : X_i \rightarrow \mathbb{C}^3/G$ related by flops in the following way

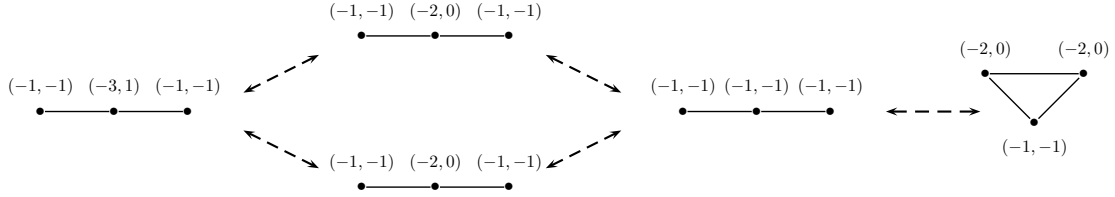


where $X_i \cong \mathcal{M}_C$ for some $C \subset \Theta$. Moreover, $X_0 \cong G\text{-Hilb}(\mathbb{C}^3)$ and every X_i is described as the union of 4 open sets isomorphic to \mathbb{C}^3 . The open covers are $X_0 = U_0 \cup U_1 \cup U_2 \cup U_3$, $X_1 = U'_0 \cup U'_1 \cup U_2 \cup U_3$, $X_2 = U_0 \cup U_1 \cup U'_2 \cup U'_3$, $X_3 = U'_0 \cup U'_1 \cup U'_2 \cup U'_3$ and $X_4 = U'_0 \cup U'_1 \cup U'_2 \cup U'_3$. See Appendix.

(2) The local coordinates in each open set are

$$\begin{array}{l} U_0 \cong \mathbb{C}_{c_2, c_3, C_1}^3 = \text{Spec}\left[\frac{\sqrt{3}f_1^2 f_3}{R_1}, \frac{-f_1 R_2}{R_1}, \frac{f_2}{f_1^2}\right] \\ U_1 \cong \mathbb{C}_{c_2, c_3, C_3}^3 = \text{Spec}\left[\frac{\sqrt{3}f_2 f_3}{R_1}, \frac{-f_2 R_2}{f_1 R_1}, \frac{f_1^2}{f_2}\right] \\ U_2 \cong \mathbb{C}_{b_2, b_3, B_3}^3 = \text{Spec}\left[\frac{\sqrt{3}f_1 f_3}{R_2}, \frac{f_1 R_1}{f_2 R_2}, \frac{f_2^2}{f_1}\right] \\ U_3 \cong \mathbb{C}_{b_2, b_3, B_1}^3 = \text{Spec}\left[\frac{\sqrt{3}f_2^2 f_3}{R_2}, \frac{f_2 R_1}{R_2}, \frac{f_1}{f_2^2}\right] \\ U'_0 \cong \mathbb{C}_{B_1, c_1, C_1}^3 = \text{Spec}\left[\frac{-\sqrt{3}f_1 f_3}{R_2}, \frac{-R_0}{\sqrt{3}f_3}, \frac{\sqrt{3}f_2 f_3}{R_1}\right] \\ U'_1 \cong \mathbb{C}_{c_3, C_1, C_3}^3 = \text{Spec}\left[\frac{R_2}{\sqrt{3}f_1 f_3}, \frac{\sqrt{3}f_3 f_3}{R_1}, \frac{\sqrt{3}f_1^2 f_3}{R_1}\right] \\ U'_2 \cong \mathbb{C}_{c_2, C_1, C_3}^3 = \text{Spec}\left[\frac{\sqrt{3}f_1 f_3}{R_2}, \frac{f_2 R_2}{f_1 R_1}, \frac{f_1 R_2}{R_1}\right] \\ U'_3 \cong \mathbb{C}_{b_2, B_1, B_3}^3 = \text{Spec}\left[\frac{-\sqrt{3}f_2 f_3}{R_1}, \frac{-f_1 R_1}{f_2 R_2}, \frac{-f_2 R_1}{R_2}\right] \\ U''_0 \cong \mathbb{C}_{b_3, B_1, B_3}^3 = \text{Spec}\left[\frac{-R_1}{\sqrt{3}f_2 f_3}, \frac{\sqrt{3}f_1 f_3}{R_2}, \frac{\sqrt{3}f_2^2 f_3}{R_2}\right] \\ U''_2 \cong \mathbb{C}_{B_1, c_2, C_1}^3 = \text{Spec}\left[\frac{-f_1 R_0}{R_2}, \frac{-\sqrt{3}f_3}{R_0}, \frac{-f_2 R_0}{R_1}\right] \end{array}$$

(3) The dual graphs of $\pi^{-1}(0)$ in each crepant resolution with the corresponding degrees for the normal bundles are:



Proof. We start by calculating explicitly $G\text{-Hilb}(\mathbb{C}^3)$ as a moduli space of representations of the McKay quiver Q with relations R .

Claim 6.6. Let $\theta \in \Theta$ be a 0-generated stability condition. Then $X_0 := G\text{-Hilb}(\mathbb{C}^3)$ is covered by U_0, U_1, U_2 and U_3 , where

$$\begin{array}{llllll} U_0 : & aB = 1 & aBbC = 1 & a = (1, 0, 0) & au = (0, 1, 0) & b = (0, 0, 1) \\ U_1 : & aB = 1 & aC = 1 & a = (1, 0, 0) & au = (0, 1, 0) & b = (0, 0, 1) \\ U_2 : & aB = 1 & aC = 1 & a = (1, 0, 0) & au = (0, 1, 0) & c = (0, 0, 1) \\ U_3 : & aCcB = 1 & aC = 1 & a = (1, 0, 0) & au = (0, 1, 0) & c = (0, 0, 1) \end{array}$$

Proof. First notice that by using the relations we have that $auB = avB = \omega^2 auB$, which implies $auB = 0$. Similarly we obtain that following paths vanish:

$$auB = avB = auC = avC = buA = bvA = buC = bvC = cuA = cvA = cuB = cvB = 0 \quad (\#)$$

We split the calculation in 5 steps:

Step1: By changing basis we can assume that $a = (1, 0, 0)$.

Step2: If $aB = aC = 0$, then it follows $au^i v^j B = au^i v^j C = 0$ by the relations of the middle vertex, which contradicts the 0-generated stability condition θ . Therefore either $aB \neq 0$ or $aC \neq 0$. Moreover we may assume that $aB = 1$ or $aC = 1$ by change of basis.

Step3: We consider the case $aB = 1$ and $aC = 0$. If $aBbC = 0$, then it turns out that any path through C is zero by the relations. This contradicts the 0-generated condition. So it must be $aBbC \neq 0$ and b not a linear multiple of a . We may assume $aBbC = 1$ and $b = (0, 0, 1)$. Next assume that $au = (\lambda, 0, \eta)$ for some $\lambda, \eta \in \mathbb{C}$. Since $auC = 0$ by $(\#)$, and $C_1 = 0, C_3 = 1$ by $aC = 0, ABbC = 1$, it follows that $\eta = 0$. Moreover since $auB = 0$ by $(\#)$, and $B_1 = 1$ by $aB = 1$, it follows that $\lambda = 0$ hence $au = 0$, which leads to $aBbC = aAaC + \omega aBbC + \omega^2 aCcC = au^2 = 0$. This contradicts $aBbC = 1$, hence au is linear independent of $(1, 0, 0)$ and $(0, 0, 1)$. Therefore we can take $au = (0, 1, 0)$ by change of basis. These are the conditions for U_0 .

Step4: The case $aB = 0$ and $aC = 1$ is similar to Step3. This case leads to U_3 .

Step5: Consider the case $aB = aC = 1$. If $au = (\lambda, 0, 0)$ for some $\lambda \in \mathbb{C}$, because $auB = 0$ and $B_1 = 1$, we must have $\lambda = 0$, hence $au = 0$. The relations $aAa + \omega aBb + \omega^2 aCc = au^2$ and $aAa + \omega^2 aBb + \omega aCc = av^2$ means

$$\begin{cases} A_1 + \omega b_1 + \omega^2 c_1 = 0 \\ \omega b_2 + \omega^2 c_2 = 0 \\ \omega b_3 + \omega^2 c_3 = 0 \end{cases} \quad \begin{cases} A_1 + \omega^2 b_1 + \omega c_1 = 0 \\ \omega^2 b_2 + \omega c_2 = 0 \\ \omega^2 b_3 + \omega c_3 = 0 \end{cases}$$

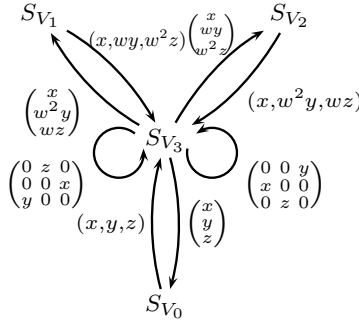
hence it follows $b_1 = c_1$ and $b_2 = c_2 = b_3 = c_3 = 0$, that is, $b = c = (b_1, 0, 0)$. This means we cannot generate the middle vertex, which contradicts the 0-generated condition. Therefore au is not a linear multiple of a , hence we can assume $au = (0, 1, 0)$.

We claim that if both of b and c are linear multiple of a and au , then it contradicts the 0-generated condition. Indeed, if we assume $b = (b_1, b_2, 0)$ and $c = (c_1, c_2, 0)$, the relations $aAa + \omega aBb + \omega^2 aCc = au^2$ and $aAa + \omega^2 aBb + \omega aCc = av^2$ are equivalent to

$$\begin{cases} A_1 + \omega b_1 + \omega^2 c_1 = u_{21} \\ \omega b_2 + \omega^2 c_2 = u_{22} \\ 0 = u_{23} \end{cases} \quad \begin{cases} A_1 + \omega^2 b_1 + \omega c_1 = v_{21} \\ \omega^2 b_2 + \omega c_2 = v_{22} \\ 0 = v_{23} \end{cases}$$

Therefore $au^2 = (0, 1, 0)u = (u_{21}, u_{22}, u_{23}) = (u_{21}, u_{22}, 0)$, $av^2 = (0, 1, 0)u = (v_{21}, v_{22}, v_{23}) = (v_{21}, v_{22}, 0)$, which are linear combinations of a and au . Therefore we can not generate the middle vertex. Consequently it must be $b = (0, 0, 1)$ or $c = (0, 0, 1)$. These conditions give U_1 and U_2 respectively. \square

The quiver structure of the CM S^G -modules S_ρ in this case is the following:



We calculate now the maps local coordinates along the exceptional curves using this quiver. For example in the open set $U_0 = \mathbb{C}_{c_2, c_3, C_1}^3$, we have that

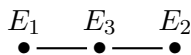
$$aC = C_1 \cdot (\text{the basis of } \rho_2)$$

which implies that $f_2 = C_1 f_1^2$, thus $C_1 = f_2 / f_1^2$. Similarly,

$$\begin{aligned} aA &= (c_1 C_1 + c_3) \cdot 1 \implies c_1 = (f_0 - c_3) f_1^2 / f_2 \\ aBbA &= (c_1 + c_3 A_1) \cdot 1 \implies f_1 f_2 = c_1 + c_3 f_0 \end{aligned}$$

which gives $c_3 = -f_1 R_2 / R_1$. Finally $auA = 3\sqrt{3}f_3 = c_2(1 - A_1 C_1) \cdot 1$, so that $c_2 = \sqrt{3}f_3 f_1^2 / R_1$. Therefore the coordinate ring of U_0 is given by $\mathbb{C}[c_2, c_3, C_1] = \mathbb{C}\left[\frac{\sqrt{3}f_3 f_1^2}{R_1}, \frac{-f_1 R_2}{R_1}, \frac{f_2}{f_1^2}\right]$. The rest of the cases are done similarly.

It follows that the fibre over the origin $\pi^{-1}(0) \subset X_0$ consists of 3 rational curves E_1 , E_2 and E_3 intersecting pairwise as



The explicit open cover shows that the curves E_1 and E_2 have degree $(-1, -1)$ while E_3 has degree $(-3, 1)$. By Lemma 8.1 we can flop E_1 and E_2 , giving rise to X_1 and X_2 respectively. By symmetry we only explain the flop of E_2 .

First Flop X_2 . In the flop of the rational curve E_2 we only need to change the open sets U_2 and U_3 . By the same method as in the dihedral case we produce the rational curve E'_2 covered by open sets U'_2 and U'_3 , both of them isomorphic to \mathbb{C}^3 , and given by

$$\begin{array}{llllll} U'_2 & b_2 = 1 & aC = 1 & a = (1, 0, 0) & au = (0, 1, 0) & c = (0, 0, 1) \\ U'_3 & b_3 = 1 & aC = 1 & a = (1, 0, 0) & au = (0, 1, 0) & c = (0, 0, 1) \end{array}$$

Second flop X_{12} . In X_1 we can flop E'_2 obtaining X_0 back, or E_1 . In the latter case we get the new curve E'_1 covered by U'_0 and U'_1 , both of the isomorphic to \mathbb{C}^3 . The conditions for the new open sets are:

$$\begin{array}{llllll} U'_0 & c_2 = 1 & aB = 1 & a = (1, 0, 0) & au = (0, 1, 0) & b = (0, 0, 1) \\ U'_1 & c_3 = 1 & aB = 1 & a = (1, 0, 0) & au = (0, 1, 0) & b = (0, 0, 1) \end{array}$$

Third flop X_{123} . The degree of the normal bundle of the curve E_3 in X_{12} is now $(-1, -1)$ so we can perform the last flop. We obtain the open sets U''_1 and U''_2 given by:

$$\begin{array}{llllll} U''_1 & c_2 = c_3 = 1 & a = (1, 0, 0) & au = (0, 1, 0) & b = (0, 0, 1) \\ U''_2 & c_1 = c_3 = 1 & a = (1, 0, 0) & au = (0, 1, 0) & b = (0, 0, 1) \end{array}$$

In the Appendix we show the representation spaces for every open set. The normal bundles of the rational curves in $\pi^{-1}(0)$ are obtain by the explicit gluings among the open sets covering the curves. These gluings are given below and the result follows.

$$\begin{aligned} U_0 \ni (c_2, c_3, C_1) &\longleftrightarrow (c_2 C_1, c_3 C_1, C_1^{-1}) \in U_1 \\ U_1 \ni (c_2, c_3, C_3) &\longleftrightarrow (-c_2 c_3^{-1}, c_3^{-1}, c_2^2(1 + c_3) - c_3^3 C_3) \in U_2 \\ U_2 \ni (b_2, b_3, B_3) &\longleftrightarrow (b_2 B_3, b_3 B_3, B_3^{-1}) \in U_3 \\ U_0 \ni (c_2, c_3, C_1) &\longleftrightarrow (c_2 C_1, c_3 C_1, C_1^{-1}) \in U_1 \\ U_1 \ni (c_2, c_3, C_3) &\longleftrightarrow (-c_2, c_3^{-1}, c_2^2(1 + c_3^{-1}) - c_3^2 C_3) \in U'_2 \\ U'_2 \ni (b_2, B_1, B_3) &\longleftrightarrow (b_2^{-1}, b_2 B_1, b_2 B_3) \in U'_3 \\ U'_0 \ni (c_3, C_1, C_3) &\longleftrightarrow (c_3^{-1}, c_3 C_1, c_3 C_3) \in U'_1 \\ U'_1 \ni (c_2, C_1, C_3) &\longleftrightarrow (-c_2 C_1, -C_1^{-1}, c_2^2 C_1(C_1 - 1) - C_1 C_3) \in U'_2 \\ U'_2 \ni (b_2, B_1, B_3) &\longleftrightarrow (b_2^{-1}, b_2 B_1, b_2 B_3) \in U'_3 \\ U'_0 \ni (c_3, C_1, C_3) &\longleftrightarrow (-c_3^{-1}, C_1 - c_3^2 C_3, C_1) \in U''_1 \\ U''_2 \ni (B_1, c_2, C_1) &\longleftrightarrow (B_1 c_2, c_2^{-1}, c_2 C_1) \in U''_1 \\ U'_3 \ni (b_3, B_1, B_3) &\longleftrightarrow (-B_1, B_1 - b_3^2 B_3, -b_3^{-1}) \in U''_1 \end{aligned}$$

□

6.4. Proof of Theorem 1.1. The proof is explicit and it follows from the direct comparison between every mutation of (Q, W) at non-trivial vertices with no loops and the description of every crepant resolution of \mathbb{C}^3/G given in Sections 5 and 6 respectively.

The case $G \cong \mathbb{Z}/n\mathbb{Z}$ is immediate: part (i) follows since the unique crepant resolution $X \cong G\text{-Hilb}(\mathbb{C}^3)$ is toric thus covered by $|G|$ open sets isomorphic to \mathbb{C}^3 (see [Nak] or [CR]), and for (ii) notice that there are no flops of X and no mutations of (Q, W) since every vertex has a loop (see 5.1).

In the rest of the cases the statement (i) follows from the part (1) in Theorems 6.1, 6.3 and 6.5. For (ii) note that for every crepant resolution $\pi : X \rightarrow \mathbb{C}^3/G$, the dual graph of the exceptional fibre $\pi^{-1}(0) = \bigcup E_i$ (described in the part (4) of Theorems 6.1, 6.3 and part (3) in Theorem 6.5)

coincide with the graph associated to the corresponding mutated quiver in Section 5 removing the trivial vertex. Recall that the *graph* of a quiver Q is obtained by forgetting the direction of the arrows. More precisely,

$$\begin{aligned} \text{For } G \cong D_{2n}, n \text{ odd:} & \quad \text{Dual graph of } \pi^{-1}(0) \subset X_i = \text{Graph of } Q_i \setminus 0 \\ \text{For } G \cong D_{2n}, n \text{ even:} & \quad \text{Dual graph of } \pi^{-1}(0) \subset X_{0\dots i}^{m\dots(m-j)} = \text{Graph of } Q_{0\dots i}^{m\dots(m-j)} \setminus 0 \\ \text{For } G \cong \mathbb{T}: & \quad \text{Dual graph of } \pi^{-1}(0) \subset X_i = \text{Graph of } Q_i \setminus 0 \end{aligned}$$

and it follows that flopping the curve E_i corresponds to mutate with respect to the vertex i .

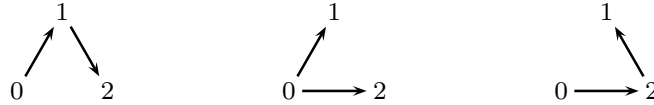
7. THE SPACE OF STABILITY CONDITIONS

Let $X \cong \mathcal{M}_C$ for some chamber $C \subset \Theta$. Given an open set $U \subset Y$ and any θ -stable representation $M \in U$ of Q for some $\theta \in C$, the explicit knowledge of the representation space of U gives every possible subrepresentation $N \subset M$. In other words, the analysis of the matrices in the representation space of every open set in an open cover of X give the inequalities defining the chamber $C \subset \Theta$. In order to do this we encode the structure of the representation space of an open set by using its *skeleton*.

Definition 7.1. Given an open set $U \subset \mathcal{M}_C$ for some $C \subset \Theta$, the *skeleton* $sk(U)$ of U is the representation of Q corresponding to the origin $\mathbf{0} \in U$.

In our situation $G \subset SO(3)$ of type $\mathbb{Z}/n\mathbb{Z}$, D_{2n} or \mathbb{T} , and take X to be a crepant resolution of \mathbb{C}^3/G . Then every open set U is isomorphic to $\mathbb{C}_{a,b,c}^3$ where a, b and c are the local coordinates of \mathbb{C}^3 . For a given point $(a, b, c) \in U$ we denote the corresponding representation by $M_{a,b,c} \in \mathcal{M}_C$. Once we choose basis for the vector spaces at every vertex of Q , the skeleton is obtained by setting $a = b = c = 0$, i.e. $sk(U) = M_{0,0,0}$.

Example 7.2. Let $G = \frac{1}{3}(1, 2, 0)$ and consider $X = G\text{-Hilb}(\mathbb{C}^3) \cong \mathcal{M}_C$. Then X is covered by 3 open sets $U_i \cong \mathbb{C}^3$ for $i = 1, 2, 3$ with skeletons:



and C is defined by $\theta_1, \theta_2 > 0$.

As a consequence of the next lemma, if there exists a finite open cover of $\mathcal{M}_C = \bigcup_{i=1}^N U_i$ and we define $C_{sk} := \{\theta \in \Theta \mid \theta(N) > 0 \text{ for every } 0 \subsetneq N \subsetneq sk(U_i) \text{ and every } i\}$, then $C = C_{sk}$.

Lemma 7.3. Let $\theta \in \Theta$ be a generic parameter. If $M_{0,0,0}$ is θ -stable then $M_{a,b,c}$ is θ -stable.

Proof. Let $(a, b, c) \in U$ and let $M_{a,b,c}$ be the corresponding representation. For every proper subrepresentation $N_{a,b,c} \subset M_{a,b,c}$, the dimension vectors for $N_{a,b,c}$ and $N_{0,0,0}$ coincide. Therefore since $N_{0,0,0} \subset M_{0,0,0}$ and $M_{0,0,0}$ is θ -stable we have that $\theta(N_{a,b,c}) = \theta(N_{0,0,0}) > 0$. \square

Theorem 7.4. (i) Let $G = D_{2n}$ with n odd and let X_i be a crepant resolution of \mathbb{C}^3/G . The chamber $C_i \subset \Theta$ for which $X_i \cong \mathcal{M}_{C_i}$ is given by the inequalities:

$$\begin{aligned} \theta_k &> 0 \text{ for } k \neq 0, 1, \\ \theta_1 &< 0, \\ \theta_1 + \theta_2 &< 0, \\ &\vdots \\ \theta_1 + \theta_2 + \dots + \theta_i &< 0, \\ \theta_1 + \theta_2 + \dots + \theta_i + \theta_{i+1} &> 0. \end{aligned}$$

The wall between C_i and C_{i+1} is defined by $\theta_1 + \theta_2 + \dots + \theta_i + \theta_{i+1} = 0$.

(ii) Let $G = D_{2n}$ with n even and let $X_{0\dots i}^{m\dots(m-j)}$ be a crepant resolution of \mathbb{C}^3/G . The chamber $C_{ij} \subset \Theta$ for which $X_{0\dots i}^{m\dots(m-j)} \cong \mathcal{M}_{C_{ij}}$ is given by the inequalities:

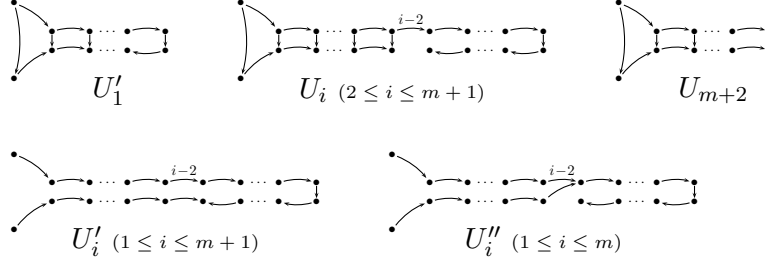
$$\left. \begin{array}{l} \theta_k > 0 \text{ for } k \neq 0, 1, m+1, \\ \sum_{k=1}^{m+1} \theta_k > 0, \\ \sum_{k=1}^m \theta_k + \theta_{m+2} > 0, \\ \theta_{m+1} + \theta_{m+2} > 0, \end{array} \right\} \begin{array}{l} \theta_1 < 0, \theta_1 + \theta_2 < 0, \dots, \sum_{k=1}^i \theta_k < 0, \\ \theta_{m+1} < 0, \theta_m + \theta_{m+1} < 0, \dots, \sum_{k=m-j+1}^{m+1} \theta_k < 0, \\ \sum_{k=1}^{i+1} \theta_k > 0, \\ \sum_{k=m-j}^{m+1} \theta_k > 0. \end{array}$$

The wall between $C_{i,j}$ and $C_{i+1,j}$ is defined by $\sum_{k=1}^{i+2} \theta_k = 0$, and the wall between $C_{i,j}$ and $C_{i,j+1}$ is given by $\sum_{k=m-j+1}^{m+1} \theta_k = 0$.

(iii) Let G be the tetrahedral group of order 12 and let X_i be a crepant resolution of \mathbb{C}^3/G . The chamber $C_i \subset \Theta$ for which $X_i \cong \mathcal{M}_{C_i}$ is given by the inequalities:

$$\begin{aligned} C_0 : & \theta_i > 0, i \neq 0 \\ C_1 : & \theta_1 < 0, \theta_2 > 0, \theta_1 + \theta_3 > 0 \\ C_2 : & \theta_1 > 0, \theta_2 < 0, \theta_2 + \theta_3 > 0 \\ C_{12} : & \theta_1 < 0, \theta_2 < 0, \theta_1 + \theta_2 + \theta_3 > 0 \\ C_{123} : & \theta_1 + \theta_3 > 0, \theta_2 + \theta_3 > 0, \theta_1 + \theta_2 + \theta_3 < 0 \end{aligned}$$

Proof. (i) Consider the open cover of X_i given in 6.1 and let $M \in X_i$ be a representation of Q . We calculate for which parameters $\theta \in \Theta_d$ the representation M is θ -stable. By the representation spaces of every open set shown in the Appendix, we can see that the skeletons for the open sets U_i , U'_i and U''_i are:



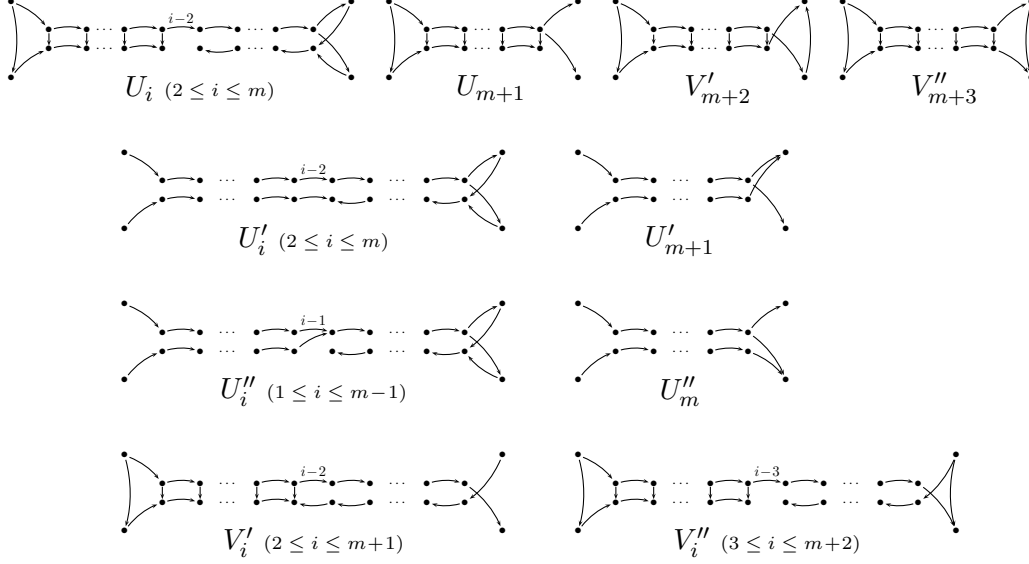
Every dot in the above picture corresponds to a basis element in the corresponding vector space in a representation of Q . Notice that the dimension vector is $\frac{1}{1}2\dots2$ so that there is one dot for each 1-dimensional vertex and two dots for each 2-dimensional vertex.

We order the subindices of the stability condition $\theta := (\theta_i)_{0 \leq i \leq m+1} \in \mathbb{Q}^{|Q_0|}$ by the sequence $\frac{0}{1}2\dots m+1$ along the vertices of Q . Let $s_i := \frac{0}{0}0\dots010\dots0$ be the dimension vector with entry 1 at the position i . With this notation, we see that every θ -stable submodule in the open sets U_1'', \dots, U_i'' contains a submodule with dimension vector s_2, \dots, s_{i+2} respectively. Similarly, there exist a submodule of dimension vector s_{i+3}, \dots, s_{m+2} in any θ -stable module contained in U_{i+2}, \dots, U_{m+2} respectively. Therefore we have that $\theta_i > 0$ for $i \geq 2$.

The rest of the condition follows by examining the remaining submodules. If $M_i \in U_i''$ then there exist a submodule $W_i \subset M_i$ with $\underline{\dim}(W_i) = \frac{0}{1}1\dots12\dots2$ where the first 2 is located in the position $i+1$. This imply that $\theta_1 + \dots + \theta_i < 0$. Finally if $N_{i+1} \in U'_{i+1}$ then there exist a submodule $V_{i+1} \subset N_{i+1}$ with $\underline{\dim}(V_{i+1}) = \frac{0}{1}1\dots10\dots0$ where the last 1 is located in the position $i+1$, which means that $\sum_{i=1}^{i+1} \theta_i > 0$.

Any other inequalities coming from the submodules of $M \in \mathcal{M}_C$ are implied by the ones we have just described, so the chamber C is defined by the inequalities of the statement. By comparing the chamber conditions of $X_{0\dots i}$ and $X_{0\dots(i+1)}$ we obtain the equation of the wall.

(ii) This time we order the subindices of the stability condition $\theta := (\theta_i)_{0 \leq i \leq m+2} \in \mathbb{Q}^{|Q_0|}$ by the sequence $\begin{smallmatrix} 0 & 2 & \dots & m \\ 1 & 2 & \dots & m+1 \end{smallmatrix}$ along the vertices of Q . The skeleton of the open sets in this case are as follows:



Consider the open cover of $X_{0..i}^{m...(m-j)}$ given in 6.3 and let $M \in X_{0..i}^{m...(m-j)}$ be a representation of Q . We now define the dimension vectors which are relevant in the proof, together with the corresponding inequality that any submodule $N \subset M$ with one of these dimension vectors produce:

Dimension vector	Inequality in Θ
$s_i := \begin{smallmatrix} 0 & & & i & & & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \end{smallmatrix}$	$\theta_i > 0$
$r_i := \begin{smallmatrix} 0 & & & i & & & 0 \\ 1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 & 0 \end{smallmatrix}$	$\sum_{k=1}^i \theta_k > 0$
$n_i := \begin{smallmatrix} 0 & & & i & & & 1 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & 0 \end{smallmatrix}$	$\sum_{k=i}^{m+1} \theta_k > 0$
$e_i := \begin{smallmatrix} 0 & & & i & & & 1 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & 1 \end{smallmatrix}$	$\sum_{k=i}^{m+2} \theta_k > 0$
$c_i := \begin{smallmatrix} 1 & & & i & & & 1 \\ 0 & 1 & \dots & 1 & 1 & 2 & \dots & 2 & 1 \end{smallmatrix}$	$\sum_{k=1}^i \theta_k < 0$
$d_i := \begin{smallmatrix} 1 & & & i & & & 0 \\ 1 & 2 & \dots & 2 & 1 & 1 & \dots & 1 & 1 \end{smallmatrix}$	$\sum_{k=i}^{m+1} \theta_k < 0$
$j := \begin{smallmatrix} 0 & & & 0 \\ 1 & 1 & \dots & 1 & 1 \end{smallmatrix}$	$\sum_{k=1}^m \theta_k + \theta_{m+2} > 0$

Note that $r_1 = s_1$, $n_{m+1} = s_{m+1}$ and $r_{m+1} = n_1$.

Lemma 7.5. (i) $\theta_i > 0$ for all $i \neq 0, 1, m+1$.

(ii) There always exists a submodule $N_1 \subset M$ with $\underline{\dim}(N_1) = n_1$.

Proof. (i) By the open covers given in Theorem 6.3 (1) and the corresponding skeletons, any crepant resolution of \mathbb{C}^3/G has at least one open set containing a submodule S_i with $\underline{\dim}(S_i) = s_i$ for $i = 2, \dots, m, m+2$. In the cases $i = 0, 1, m+1$ note that S_0 do not belong to any open set so that there's no condition of the form $\theta_0 > 0$. The submodule S_1 is only contained in U'_1 and U_2 , which implies that only $X_{0..i}^{m...(m-j)}$ for any j have the condition $\theta_1 > 0$. Finally, only U_{m+1} and V'_{m+2} contain the submodule S_{m+1} , so that the condition $\theta_{m+1} > 0$ only is valid in $X_{0..k}$ for any k .

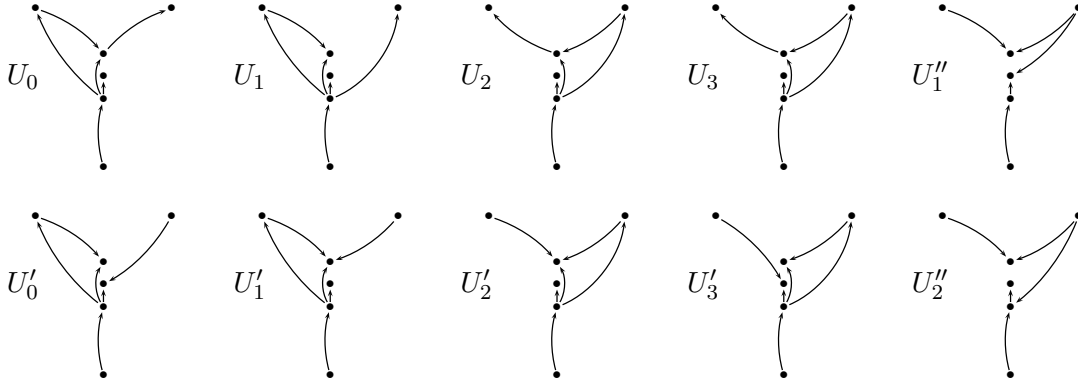
(ii) Notice that for every k the submodule $N_1 \in U'_k, V'_k$, and every $X_{0..i}^{m...(m-j)}$ contains at least one of these affine sets. \square

Therefore the dimension vectors that we have to consider in every open set are the following:

Open set	Dimension vectors	Open set	Dimension vectors
$(2i \leq m), U_i$	$s_{i-1}, s_i, r_{i-1}, n_i$	$(i \leq m-1), U_i''$	$s_{i+1}, n_{i+1}, c_1, \dots, c_i$
U_{m+1}	$s_m, s_{m+1}, s_{m+2}, r_m$	U_m''	s_{m+1}, s_{m+2}, j, c_m
V'_{m+2}	s_{m+1}, n_1	V'_i	$s_{i-1}, s_{m+2}, r_{i-1}, n_{i-1}, n_1, j, d_i, \dots, d_{m+1}$
V''_{m+3}	s_{m+2}, j, e_{m+1}	$(i \leq m+1), V_i''$	$s_{i-2}, s_{i-1}, s_{m+2}, r_{i-2}, e_{i-1}, d_{i-1}, \dots, d_{m+1}$
$(1i \leq m), U'_i$	$s_i, r_i, n_i, n_1, c_1, \dots, c_{i-1}$	V''_{m+2}	$s_m, s_{m+2}, r_m, e_m, d_{m+1}$
U'_{m+1}	$s_{m+1}, s_{m+2}, n_m, n_1, c_m$		

The result follows by going through the open cover of $X_{0..i}^{m..(m-j)} \cong \mathcal{M}_{C_{ij}}$ given in Theorem 6.3, and writing down the corresponding inequalities.

(iii) The skeletons in this case are:



As expected, only the skeletons for $U_i, i = 0, \dots, 3$ are generated from the vertex 0. Indeed, this is equivalent to the 0-generated stability condition which only $G\text{-Hilb}(\mathbb{C}^3)$ satisfies.

Now take the open covers of X_i given in Theorem 6.5. Then the inequalities defining the chambers C_i for which $X_i \cong \mathcal{M}_{C_i}$ are given by the submodules of the above skeletons, and the result follows. \square

Remark 7.6. The set of inequalities in Theorem 7.4 does not give the reduce description of the chamber $C \subset \Theta$. Nevertheless, it can be shown that for any crepant resolution the reduce number of walls or inequalities defining C is precisely $|Q_0| - 1$, which coincides with the number of components of the fibre over the origin (or the number of non-trivial irreducible representations of G).

Corollary 7.7. There exists a connected region $F \subset \Theta_{\mathbf{d}}$ where every crepant resolution of \mathbb{C}^3/G can be found and such that every wall crossing in F corresponds to a flop.

7.1. Stability conditions and mutations. In this section we compare the classical approach of changing the stability condition on the representations of the McKay quiver to obtain all crepant resolutions of \mathbb{C}^3/G , and the mutation approach.

Let $G \subset SO(3)$ of type $\mathbb{Z}/n\mathbb{Z}$, D_{2n} or \mathbb{T} , and let (Q, W) be the McKay QP. Let \mathbb{Z}^{Q_0} be a space of dimension vectors, and we take a canonical basis $\{\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n\}$. Let $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{Q_0}, \mathbb{Z})$ be the dual space with the dual basis $\mathbf{e}_0^*, \mathbf{e}_1^*, \dots, \mathbf{e}_n^*$ and define $\Theta := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{Q_0}, \mathbb{Z}) \otimes \mathbb{Q}$ the whole parameter space. Let $\mathbf{d} = \sum_{i \in Q_0} (\dim \rho_i) \mathbf{e}_i$ with $\rho_i \in \text{Irr } G$.

Let $\mu(Q, W)$ a QP obtained by a sequence of mutations $\mu = \mu_{i_1} \cdots \mu_{i_m}$ from the McKay QP. We denote by $\Lambda = \mathcal{P}(\mu(Q, W))$ the Jacobian algebra. We note that Λ is a 3-Calabi-Yau (3-CY for short) algebra since $S * G$ is 3-CY and the property of 3-CY is closed under Morita equivalences

and mutations. We fix a vertex $i \in Q_0$ with no loops and let P_i be the projective Λ -module and S_i the simple module associated to the vertex i . Then, there is an exact sequence

$$(7.1) \quad 0 \rightarrow P_i \rightarrow X_2 = \bigoplus_{a \in Q_1, ha=i} P_{ta} \rightarrow X_1 = \bigoplus_{a \in Q_1, ta=i} P_{ha} \xrightarrow{f} P_i \rightarrow S_i \rightarrow 0.$$

Let $(-, -)$ be a symmetric bilinear form on \mathbb{Z}^{Q_0} defined by

$$(\mathbf{e}_i, \mathbf{e}_j) = \begin{cases} 2 & i = j \\ -\#(i \rightarrow j) & i \neq j \end{cases}$$

In our case, if i and j are adjacent, then we can see that there is only one arrow from $i \rightarrow j$, so $(\mathbf{e}_i, \mathbf{e}_j)$ is -1 .

We define $(M, N) := (\dim M, \dim N) := (\dim M, \dim N)$ for any finite dimensional Λ -modules M, N . We denote by s_i the reflection with respect to a vertex i , which is defined by

$$s_i \alpha := \alpha - (\alpha, \mathbf{e}_i) \mathbf{e}_i$$

for any dimension vector $\alpha \in \mathbb{Z}^{Q_0}$ and dually

$$s_i \theta := \theta - \theta_i \sum_{j=0}^n (\mathbf{e}_i, \mathbf{e}_j) \mathbf{e}_j^*.$$

Trivially, for any dimension vector α , $\theta(\alpha) = 0$ if and only if $(s_i \theta)(s_i \alpha) = 0$. For a sequence $\mu = \mu_{i_1} \cdots \mu_{i_m}$ of mutations, we consider the corresponding sequence of reflections $\omega = s_{i_1} \cdots s_{i_m}$. Then dimension vectors $\omega \mathbf{d}$ determine parameter spaces $\Theta_{\omega \mathbf{d}}$. Let $\theta^0 \in \Theta_{\omega \mathbf{d}}$ be the 0-generated stability condition and C_0 the chamber in $\Theta_{\omega \mathbf{d}}$ defined by the inequalities of $\theta_i^0 > 0$ for $i \neq 0$.

Lemma 7.8. The chamber of $\mathcal{M}_{\theta^0, \omega \mathbf{d}}(\Lambda)$ is C_0 .

Proof. It follows from direct calculations. We see that all simple modules associated to vertices can be a subrepresentation of some point in $\mathcal{M}_{\theta^0, \omega \mathbf{d}}(\Lambda)$ \square

Note that we know there is a one to one correspondence between flops of $G\text{-Hilb}(\mathbb{C}^3)$ and mutations of the McKay QP. The goal of this subsection is the next result.

Theorem 7.9. Let $X \rightarrow \mathbb{C}^3/G$ be an arbitrary crepant resolution. Then $X \cong \mathcal{M}_{\theta^0, \omega \mathbf{d}}(\Gamma)$ for the corresponding Jacobian algebra $\Gamma = \mathcal{P}(\mu(Q, W))$. Moreover, there exists a corresponding sequence of wall crossings from $G\text{-Hilb}(\mathbb{C}^3)$ which leads to $X \cong \mathcal{M}_{\theta, \mathbf{d}}(\Lambda)$ where $\Lambda = \mathcal{P}(Q, W)$ and the chamber $C \subset \Theta_{\mathbf{d}}$ containing θ is given by the inequalities $\theta(\omega^{-1} \mathbf{e}_i) > 0$ for any $i \neq 0$.

The rest of the section is dedicated to prove the above theorem.

Definition 7.10. For any parameter $\theta \in \Theta$, we define the full subcategory $\mathcal{S}_\theta(\Lambda)$ of $\text{Mod } \Lambda$ consisting of θ -semistable finite dimensional Λ -modules. Moreover we denote by $\mathcal{S}_{\theta, \alpha}(\Lambda)$ the full subcategory of $\mathcal{S}_\theta(\Lambda)$ consisting of θ -semistable Λ -modules of dimension vector α if $\mathcal{S}_{\theta, \alpha}(\Lambda)$ is not empty.

In the exact sequence (7.1), let K_i be the kernel of f fitting in the exact sequence

$$(7.2) \quad 0 \rightarrow K_i \xrightarrow{g} X_1 \xrightarrow{f} P_i \rightarrow S_i \rightarrow 0.$$

Then it can be checked that $T_i := \Lambda/P_i \oplus K_i$ is a tilting Λ -module of projective dimension one. We put $\Gamma = \text{End}_\Lambda(T_i)$. By a similar strategy in [BIRS], we can prove that $\Gamma \simeq \mathcal{P}(\mu_i \omega(Q, W))$.

Lemma 7.11. Let M be a finite dimensional Λ -module of dimension vector $\alpha = (\alpha_k)$. Then the alternating sum of the dimension vector of $\mathbb{R}\text{Hom}_\Lambda(T_i, M)$ is given by the following formula:

$$\underline{\dim}_\Gamma \text{Hom}_\Lambda(T_i, M) - \underline{\dim}_\Gamma \text{Ext}_\Lambda^1(T_i, M) = s_i \alpha.$$

Proof. For each $j \in Q_0$, e_j denotes the corresponding idempotent of Λ . The following hold:

$$\mathrm{Hom}_\Lambda(T_i, M)e_j \simeq \mathrm{Hom}_\Lambda(e_j T_i, M) = \begin{cases} \mathrm{Hom}_\Lambda(K_i, M) & \text{if } j = i \\ \mathrm{Hom}_\Lambda(P_j, M) & \text{if } j \neq i \end{cases}$$

and

$$\mathrm{Ext}_\Lambda^1(T_i, M)e_j \simeq \mathrm{Ext}_\Lambda^1(e_j T_i, M) = \begin{cases} \mathrm{Ext}_\Lambda^1(K_i, M) & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

By applying $\mathrm{Hom}_\Lambda(-, M)$ to the exact sequence $0 \rightarrow P_i \rightarrow X_2 \rightarrow K_i \rightarrow 0$, we have

$$0 \rightarrow \mathrm{Hom}_\Lambda(K_i, M) \rightarrow \mathrm{Hom}_\Lambda(X_2, M) \rightarrow \mathrm{Hom}_\Lambda(P_i, M) \rightarrow \mathrm{Ext}_\Lambda^1(K_i, M) \rightarrow 0.$$

Hence we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_\Lambda(K_i, M) - \dim_{\mathbb{C}} \mathrm{Ext}_\Lambda^1(K_i, M) = \dim_{\mathbb{C}} \mathrm{Hom}_\Lambda(X_2, M) - \dim_{\mathbb{C}} \mathrm{Hom}_\Lambda(P_i, M).$$

so that $\underline{\dim}_\Gamma \mathrm{Hom}_\Lambda(T_i, M) - \underline{\dim}_\Gamma \mathrm{Ext}_\Lambda^1(T_i, M)$ is equal to

$$\begin{aligned} & \sum_{j \neq i} \alpha_j \mathbf{e}_j + (\dim_{\mathbb{C}} \mathrm{Hom}_\Lambda(K_i, M) - \dim_{\mathbb{C}} \mathrm{Ext}_\Lambda^1(K_i, M)) \mathbf{e}_i \\ &= \sum_{j \neq i} \alpha_j \mathbf{e}_j + \left(\sum_{a \in Q_1, ha=i} \alpha_{ta} - \alpha_i \right) \mathbf{e}_i = \alpha - (2\alpha_i - \sum_{a \in Q_1, ha=i} \alpha_{ta}) \mathbf{e}_i = \alpha - (\alpha, \mathbf{e}_i) \mathbf{e}_i. \end{aligned}$$

□

We have a similar result as in the two dimensional case treated in [SY].

Theorem 7.12. If $\theta_i > 0$, then there is an equivalence

$$\mathcal{S}_\theta(\Lambda) \xrightleftharpoons[-\otimes_\Gamma T_i]{\mathrm{Hom}_\Lambda(T_i, -)} \mathcal{S}_{s_i \theta}(\Gamma)$$

of categories which preserves S -equivalence classes. Moreover this equivalence induces an isomorphism

$$\mathcal{M}_{\theta, \alpha}(\Lambda) \cong \mathcal{M}_{s_i \theta, s_i \alpha}(\Gamma)$$

of varieties for any $\alpha \in \mathbb{Z}^{Q_0}$.

Proof. Since T_i is a tilting module, there is a derived equivalence

$$\mathcal{D}(\mathrm{Mod} \Lambda) \xrightleftharpoons[-\otimes_\Gamma T_i]{\mathbb{R}\mathrm{Hom}_\Lambda(T_i, -)} \mathcal{D}(\mathrm{Mod} \Gamma).$$

The functor $\mathbb{R}\mathrm{Hom}_\Lambda(T_i, -)$ induces a functor $\mathrm{Hom}_\Lambda(T_i, -)$ from $\mathcal{S}_\theta(\Lambda)$ to $\mathrm{mod} \Gamma$. We show that $\mathrm{Hom}_\Lambda(T_i, -)$ is well-defined. Let M be a θ -semistable Λ -module of dimension α . By applying $\mathrm{Hom}_\Lambda(-, M)$ to the exact sequence (7.2) and using the fact that Λ is 3-CY, we have

$$\mathrm{Ext}_\Lambda^1(T_i, M) \simeq \mathrm{Ext}^3(S_i, M) \simeq D \mathrm{Hom}_\Lambda(M, S_i).$$

Since $\theta_i > 0$, M doesn't have S_i as a factor. So we have $\mathrm{Hom}_\Lambda(M, S_i) = 0$, hence $\mathrm{Ext}_\Lambda^1(T_i, M) = 0$. Next we show that $M' = \mathrm{Hom}_\Lambda(T_i, M)$ is $s_i \theta$ -semistable. By Lemma 7.11 we have

$$(s_i \theta)(M') = (s_i \theta)(\underline{\dim}_\Gamma \mathrm{Hom}_\Lambda(T_i, M)) = (s_i \theta)(s_i \alpha) = \theta(\alpha) = 0.$$

We take any proper submodule N' of M' and consider the following exact sequence

$$0 \rightarrow N' \rightarrow M' \rightarrow C \rightarrow 0.$$

By applying $-\otimes_\Gamma T_i$ to the above, since $\mathrm{Tor}_1^\Gamma(M', T_i) = 0$ we have an exact sequence

$$0 \rightarrow \mathrm{Tor}_1^\Gamma(C, T_i) \rightarrow N' \otimes_\Gamma T_i \xrightarrow{f} M' \otimes_\Gamma T_i \simeq M.$$

We have $\text{Tor}_1^\Gamma(N, T_i)e_j = \text{Tor}_1^\Gamma(N, T_i e_j) = 0$ since $T_i e_j \simeq \text{Hom}_\Lambda(\Lambda, T_i)e_j \simeq \text{Hom}_\Lambda(P_j, T_i)$ is a projective Γ^{op} -module. So $\underline{\dim}_\Gamma \text{Tor}_1^\Gamma(C, T_i) = \mathbf{e}_i^m$ for some non-negative integer m . Since $\text{Tor}_1^\Gamma(N', T_i) = 0$ we have $\underline{\dim}_\Lambda N' \otimes_\Gamma T_i = s_i \underline{\dim}_\Gamma N'$. Thus since $\text{Im } f$ is a submodule of M , we have

$$\begin{aligned} (s_i \theta)(N') &= (s_i \theta)(s_i \underline{\dim}_\Lambda N' \otimes_\Gamma T_i) \\ &= \theta(\underline{\dim}_\Lambda N' \otimes_\Gamma T_i) \\ &= \theta(\mathbf{e}_i^m) + \theta(\text{Im } f) \geq 0. \end{aligned}$$

Note that if M is θ -stable, then M' is also $s_i \theta$ -stable since $\theta(\text{Im } f) > 0$.

The converse is proved similarly. Moreover one can easily check that S -equivalent classes are preserved.

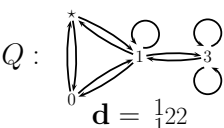
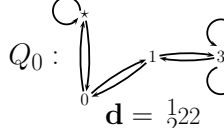
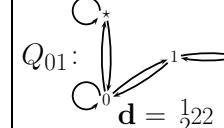
Furthermore, for any dimension vector α , the functors $\text{Hom}_\Lambda(T_i, -)$ and $- \otimes_\Gamma T_i$ induce inverse morphisms $f : \mathcal{M}_{\theta, \alpha}(\Lambda) \rightarrow \mathcal{M}_{s_i \theta, s_i \alpha}(\Gamma)$ and $g : \mathcal{M}_{s_i \theta, s_i \alpha}(\Gamma) \rightarrow \mathcal{M}_{\theta, \alpha}(\Lambda)$ so that there is an isomorphism between the moduli spaces (see [SY, Theorem 5.6] for details). \square

Corollary 7.13. A chamber $C \subset \Theta_{\omega \mathbf{d}}$ is mapped to a chamber $s_i C \subset \Theta_{s_i \omega \mathbf{d}}$. More precisely, if C is defined by inequalities $\theta(\alpha) > 0$ for a set of vectors $\{\alpha\}$, $s_i C$ is defined by inequalities $\theta(s_i \alpha) > 0$.

Proof. For any $\theta \in \Theta_{\omega \mathbf{d}}$ by Theorem 7.12, a Λ -module M of dimension vector $\omega \mathbf{d}$ is θ -(semi)stable if and only if $\text{Hom}_\Lambda(T_i, M)$ is $s_i \theta$ -(semi)stable, so the first assertion follows. The second assertion follows from the fact that $s_i \theta(s_i \alpha) > 0$ is equivalent to $\theta(\alpha) > 0$ for any $\theta \in C$. \square

Proof of Theorem 7.9. Let X be any crepant resolution and $\mu(Q, W)$ the corresponding QP given in the previous sections. By Theorem 7.12 it follows that $\mathcal{M}_{\theta^0, \omega \mathbf{d}}(\mu(Q, W)) \cong \mathcal{M}_{\omega^{-1} \theta^0, \mathbf{d}}(Q, W)$ and it can be checked that $X \simeq \mathcal{M}_{\omega^{-1} \theta^0, \mathbf{d}}(Q, W)$. Also by combining Lemma 7.8 and Corollary 7.13, the chamber containing $\omega^{-1} \theta^0$ is given by the equalities $\theta(\omega^{-1} \mathbf{e}_i) > 0$. \square

Example 7.14. Let $G = D_{14}$. The rows in the following diagram correspond to the different chambers in the three mutated algebras for which the crepant resolution of \mathbb{C}^3/G shown in the left column can be realized. Note that in any mutated algebra we can find the corresponding crepant resolution in the chamber containing the 0-generated parameter.

	$Q :$  $\mathbf{d} = \frac{1}{2} \mathbf{1}_{22}$	$Q_0 :$  $\mathbf{d} = \frac{1}{2} \mathbf{1}_{22}$	$Q_{01} :$  $\mathbf{d} = \frac{1}{2} \mathbf{1}_{22}$
$G\text{-Hilb}(\mathbb{C}^3)$ $\underbrace{(-1, -1)} \quad \underbrace{(-2, 0)} \quad \underbrace{(-3, 1)}$	$\theta_0 > 0$ $\theta_1 > 0$ $\theta_2 > 0$	$\theta_0 < 0$ $\theta_0 + \theta_1 > 0$ $\theta_2 > 0$	$\theta_0 + \theta_1 < 0$ $\theta_0 > 0$ $\theta_1 + \theta_2 > 0$
X_0 $\underbrace{(-1, -1)} \quad \underbrace{(-1, -1)} \quad \underbrace{(-3, 1)}$	$\theta_0 < 0$ $\theta_0 + \theta_1 > 0$ $\theta_2 > 0$	$\theta_0 > 0$ $\theta_1 > 0$ $\theta_2 > 0$	$\theta_0 + \theta_1 > 0$ $\theta_1 < 0$ $\theta_1 + \theta_2 > 0$
X_{01} $\underbrace{(-2, 0)} \quad \underbrace{(-1, -1)} \quad \underbrace{(-2, 0)}$	$\theta_1 > 0$ $\theta_0 + \theta_1 < 0$ $\theta_0 + \theta_1 + \theta_2 > 0$	$\theta_0 + \theta_1 > 0$ $\theta_1 < 0$ $\theta_1 + \theta_2 > 0$	$\theta_0 > 0$ $\theta_1 > 0$ $\theta_2 > 0$

8. FLOPPABLE CURVES IN \mathcal{M}_G

Let $G \subset SO(3)$ of type $\mathbb{Z}/n\mathbb{Z}$, D_{2n} or \mathbb{T} . Let $\pi : X \rightarrow \mathbb{C}^3/G$ be a crepant resolution and $E \subset X$ be a rational curve. In this section we prove that the only rational curves in a crepant resolution of \mathbb{C}^3/G that can be flopped are the $(-1, -1)$ -curves.

There are three possible degrees for the normal bundle $\mathcal{N}_{X|E}$ over a curve $E \cong \mathbb{P}^1$ in X , namely $(-1, -1)$, $(-2, 0)$ and $(-3, 1)$, and all three types appear in the families treated in this paper. For every $(-1, -1)$ -curve there always exists a flop $X \dashrightarrow X'$ of E where X and X' are isomorphic in codimension one. If E is a $(-2, 0)$ -curve then we can use the *width of E* defined in [Pagoda] to conclude that E is always contained on a scroll, which implies that it does not exist a small contraction of X which contracts E .

There are only two $(-3, 1)$ -curves: $E_m \subset D_{2n}\text{-Hilb}(\mathbb{C}^3)$ when n is odd and $E_2 \subset \mathbb{T}\text{-Hilb}(\mathbb{C}^3)$. In both cases we use the fact that $X \cong \mathcal{M}_C$ for some chamber $C \in \Theta$ and we consider the contraction of E as the map $\mathcal{M}_C \rightarrow \mathcal{M}_{\bar{\theta}}$ where $\bar{\theta} \in \bar{C}$ lies on a wall of the chamber C (cf. [CI] §3.2). By the study of S -equivalence classes we are able to describe explicitly the contracted locus and conclude that such a contraction is divisorial, i.e. the curve is not floppable.

We finish the section giving an alternative proof of the fact that $E_2 \subset \mathbb{T}\text{-Hilb}(\mathbb{C}^3)$ is not floppable using *contraction algebras*.

Lemma 8.1. Let $G \subset SO(3)$ of type $\mathbb{Z}/n\mathbb{Z}$, D_{2n} or \mathbb{T} , and let $\pi : X \rightarrow \mathbb{C}^3/G$ be a crepant resolution. Then only the rational curves $E \subset X$ with degree of normal bundle $(-1, -1)$ are floppable.

Proof. Let (Q, R) the McKay quiver with relations, $\Lambda = \mathbb{C}Q/R$, $\mathbf{d} := (\dim \rho)_{\rho \in \text{Irr } G}$ and $\theta \in \Theta$ be the 0-generated parameter. Denote by $\mathcal{M}_{\theta} := \mathcal{M}_{\theta, \mathbf{d}}(\Lambda)$. By the part (3) in Theorems 6.1, 6.3 and 6.5 only $E_m \subset D_{2n}\text{-Hilb}(\mathbb{C}^3)$ when n is odd and $E_2 \subset \mathbb{T}\text{-Hilb}(\mathbb{C}^3)$ are $(-3, 1)$ -curves. Since the open sets covering these curves do not change under the flop of any other curve, it is enough to prove that they are not floppable in $G\text{-Hilb}(\mathbb{C}^3)$. Thus, it is enough to show the following three claims:

- (i) If E is a $(-2, 0)$ -curve then E is contained on a scroll.
- (ii) Let $G = D_{2n} \subset SO(3)$ with n odd. Then the $(-3, 1)$ -curve on \mathcal{M}_{θ} is not floppable.
- (iii) Let $G \subset SO(3)$ be the tetrahedral group. Then the $(-3, 1)$ -curve on \mathcal{M}_{θ} is not floppable.

Proof of (i). By the covering of X given in Section 6 we know that $E \subset X$ is covered by two open sets U and U' where $U, U' \cong \mathbb{C}^3$. First notice that for every curve E of type $(-2, 0)$ we can make a suitable change of basis on U or U' to obtain the gluing to be of the form $U \setminus \{a = 0\} \ni (a, b, c) \mapsto (a^{-1}, a^2b, c) \in U' \setminus \{a' = 0\}$. It is straightforward in most cases, although we give here some of them:

In D_{2n} with n odd have $U'_{m+1} \setminus \{a = 0\} \ni (a, b, B) \mapsto (a^2(d^4 - D^2/4), d, a^{-1}) \in U_{m+2} \setminus \{u = 0\}$, so we can change of coordinates in U'_{m+1} by $(\bar{a}, \bar{d}, \bar{D}) = (a, d, d^2 - D)$.

In D_{2n} with n even have $V''_{m+1} \setminus \{d = 0\} \ni (d, D, C') \mapsto (C'^2 + d^2D, d^{-1}, C') \in V''_{m+2} \setminus \{c' = 0\}$, so we can change of coordinates in V''_{m+2} by $(\bar{A}, \bar{c}', \bar{C}') = (A - C'^2, c', C')$.

In \mathbb{T} have $U_1 \setminus \{c_3 = 0\} \ni (c_2, c_3, C_3) \mapsto (-c_2, c_3^{-1}, c_2^2(1 + c_3^{-1}) - c_3^2C_3) \in U'_2 \setminus \{B_1\}$, so we can change of coordinates in U'_2 by $(\bar{b}_1, \bar{B}_1, \bar{B}_3) = (b_1, B_1, b_1^2(1 + B_1) - B_3)$.

The *width* of a $(-2, 0)$ -curve $E \in X$ is defined in [Pagoda] as

$$n := \sup\{n \mid \exists \text{ scheme } E_n \text{ with } E \subset E_n \subset X \text{ s.t. } E_n \cong E \times \text{Spec } \mathbb{C}[\varepsilon]/\varepsilon^n\}$$

Once we have the gluing in the form $(a, b, c) \mapsto (a^{-1}, a^2b, c)$, the curve $E \subset U$ is defined by the ideal $I = (b, c)$ and for any $k > 0$ the ideal $J_k = (b, c^k)$ satisfy the conditions of the criteria in Proposition 5.10 in [Pagoda], so the curve E has infinity width. Thus E moves in a scroll $S \subset X$ so there is no small contraction of X which contracts only E .

Proof of (ii). First we note that, if $\bar{\theta}$ is a parameter with $\theta_i > 0$ for $i \neq 0, m$ and $\theta_m = 0$, then there is a morphism $f : \mathcal{M}_{\theta} \rightarrow \mathcal{M}_{\bar{\theta}}$ which cannot be further factored into birational morphisms between normal varieties. If M is a point on \mathcal{M}_{θ} , then the image $[M] := f(M)$ is an S -equivalence class of M with respect to $\bar{\theta}$.

In the open cover U_{m+2} , put $x = u, y = v, z = V$. We consider the hypersurface $X \subset U_{m+2}$ defined by $y^2 - xz^2 = 0$. We prove that the surface X is contracted to a curve by calculating

$$U_{m+2} \simeq \mathbb{C}_{\alpha,y,z}^3 \ni M = \begin{array}{c} \mathbb{C} \\ \downarrow (1,0) \quad \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix} \\ \downarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \alpha \quad \mathbb{C}^2 \\ \uparrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \uparrow (0,1) \quad \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix} \\ \mathbb{C} \end{array} \quad \begin{array}{c} u_m = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ v = \begin{pmatrix} y & z \\ -\alpha z & -y \end{pmatrix} \end{array}$$
$$0 \subsetneq M'' \subsetneq M' \subsetneq M.$$
$$[M'/M''] \simeq [\sqrt{\alpha} = u_m \circlearrowleft \mathbb{C} \circlearrowright v = 0], [M''] \simeq [0 = u_m \circlearrowleft \mathbb{C} \circlearrowright v = 0]$$
$$a = (1, 0, 0), b = (0, 0, 1), c = (y^2 - \alpha z^2, y, z), A = \alpha \begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix}, C = \begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix},$$

$$u = \begin{pmatrix} 0 & \omega^2 y & \omega^2 z \\ \alpha z - \omega(y^2 - \alpha z^2) & \omega^2 y & \omega^2 z \\ \omega^2 \alpha y & -\omega^2 \alpha z & -\omega^2 y \end{pmatrix}, v = \begin{pmatrix} 0 & \omega y & \omega z \\ \alpha z - \omega^2(y^2 - \alpha z^2) & \omega y & \omega^2 z \\ \omega \alpha y & -\omega \alpha z & -\omega y \end{pmatrix}.$$

$\omega^2 \begin{pmatrix} -y & \alpha(z+\omega^2) \\ -(z+\omega^2) & y \end{pmatrix} = u \bigcirc \mathbb{C}^2 \bigcirc v = \omega \begin{pmatrix} -y & \alpha(z+\omega^2) \\ -(z+\omega) & y \end{pmatrix}$. The eigenvalues of u, v are 0 and the eigenvectors are $(y, z + \omega^2), (\alpha(z + \omega^2), y)$ and $(y, z + \omega), (\alpha(z + \omega), y)$ respectively. Here we put $X_\alpha^+ = X_\alpha \cap (y = \sqrt{\alpha}(z + w))$ and $X_\alpha^- = X_\alpha \cap (y = -\sqrt{\alpha}(z + w))$. Then if M is on X_α^+ , all eigenvectors are multiples of the vector $(\sqrt{\alpha}, 1)$. So by taking the subrepresentation M''^+ of M' generated by $(\sqrt{\alpha}, 1)$, we have a filtration of θ -semistable representations:

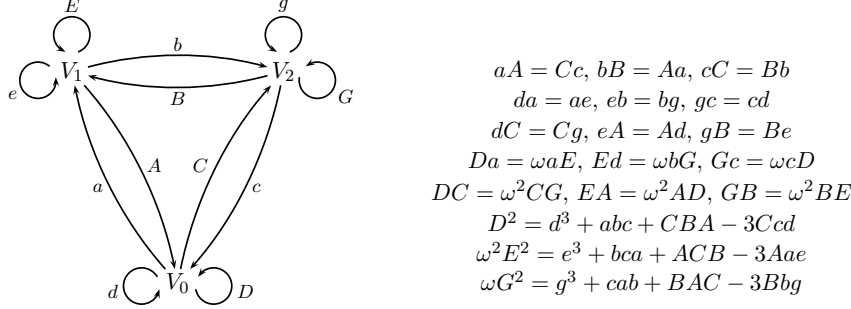
$$0 \subsetneq M'' \subsetneq M' \subsetneq M$$

8.1. A contraction algebra. In this section we give an alternative proof of Lemma 8.1 by using *contraction algebras*. Let G be the tetrahedral group \mathbb{T} of order 12, $S := \mathbb{C}[x, y, z]$ as usual. Let $M_i := (S \otimes \rho)^G$ for $i = 0, \dots, 3$ be the non-isomorphic CM S^G -modules with $M_0 \cong R$, and

let $M := \bigoplus_{i=0}^3 M_i$. Then the algebra $\Lambda := \text{End}_{SG}(M)$ is isomorphic to the Jacobian algebra $\mathcal{P}(Q, W)$ for the McKay QP (Q, W) given in the previous sections.

Let $X := \mathbb{T}\text{-Hilb}(\mathbb{C}^3) \cong \mathcal{M}_{\theta^0, \mathbf{d}}(\Lambda)$ with $\theta^0 = (-5, 1, 1, 1)$ and $\mathbf{d} = (1, 1, 1, 3)$. Let $E_3 \subset X$ the rational curve of type $(-3, 1)$, which corresponds with the vertex $3 \in Q_0$. If E_3 is floppable there exists a small contraction $\tau : X \rightarrow Y$, where we can realize Y as the moduli space $\mathcal{M}_{\overline{\theta^0}, \overline{\mathbf{d}}}(\Gamma)$ with $\overline{\theta^0} = (-3, 1, 1)$, $\overline{\mathbf{d}} = (1, 1, 1)$ and $\Gamma := \text{End}_{SG}(M/M_3)$ is obtained by removing the module M_3 from M . We call Γ a *contraction algebra* by its analogy with the geometry.

It turns out that Γ is isomorphic to the path algebra $\mathbb{C}\overline{Q}/\overline{R}$ for the following quiver $(\overline{Q}, \overline{R})$ with relations:



Thus we obtain $Y \cong \mathcal{M}_{\overline{\theta^0}, \overline{\mathbf{d}}}(\overline{Q}, \overline{R}) = U_1 \cup U_2 \cup U_3$ where U_i are hypersurfaces given by equations:

$$\begin{aligned}
 U_1 : (wG^2 &= d^3 + c + c^2C^3 - 3cCd) \subset \mathbb{C}_{c,C,d,G}^4 \\
 U_2 : (wG^2 &= d^3 + b^2B + bB^2 - 3bBd) \subset \mathbb{C}_{c,B,d,G}^4 \\
 U_3 : (wG^2 &= d^3 + A + A^2a^3 - 3aAd) \subset \mathbb{C}_{c,C,d,G}^4
 \end{aligned}$$

Therefore Y has a singular line L which in U_2 is given by the points $(d, d, d, 0)$. It can be checked that as in Lemma 8.1 that the preimage of L is precisely the equation $y^2 = xz^2 - xz + x$ by setting $C_3 = x$, $c_2 = y$ and $c_1 = y^2 - xz^2$. Therefore τ is not a small contraction, therefore E_3 is not floppable. Moreover, this construction coincides with the contraction map $\mathcal{M}_{\theta^0} \rightarrow \mathcal{M}_{\overline{\theta^0}}$ described in the previous lemma where $\overline{\theta^0}$ is a stability condition at the wall $\theta_3 = 0$.

9. APPENDIX

Definition 9.1. Let $G = D_{2n}$, $\alpha := m - i$.

FIGURE 5. Open sets for crepant resolutions of \mathbb{C}^3/G for $G = D_{2n}$, n odd.

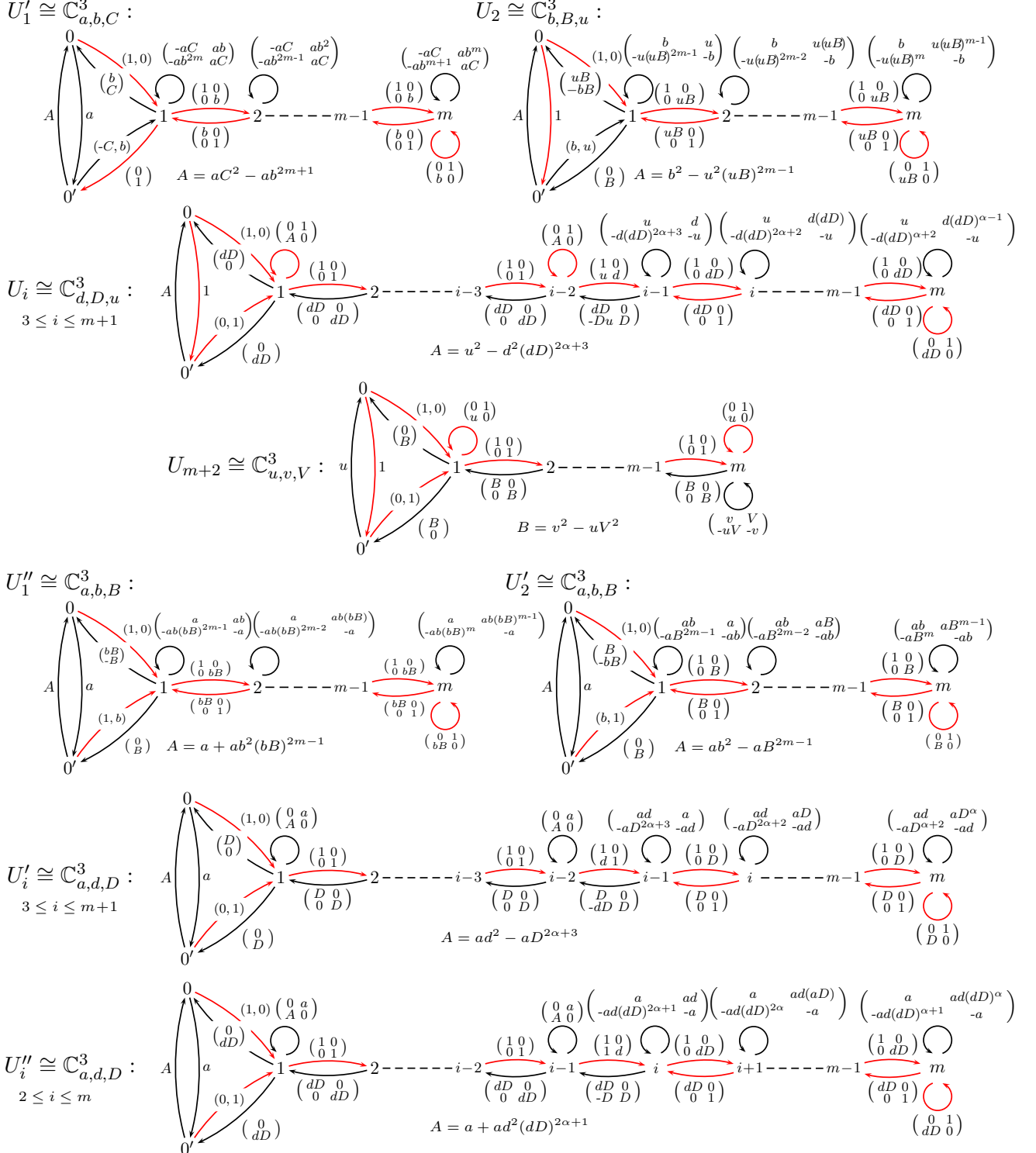


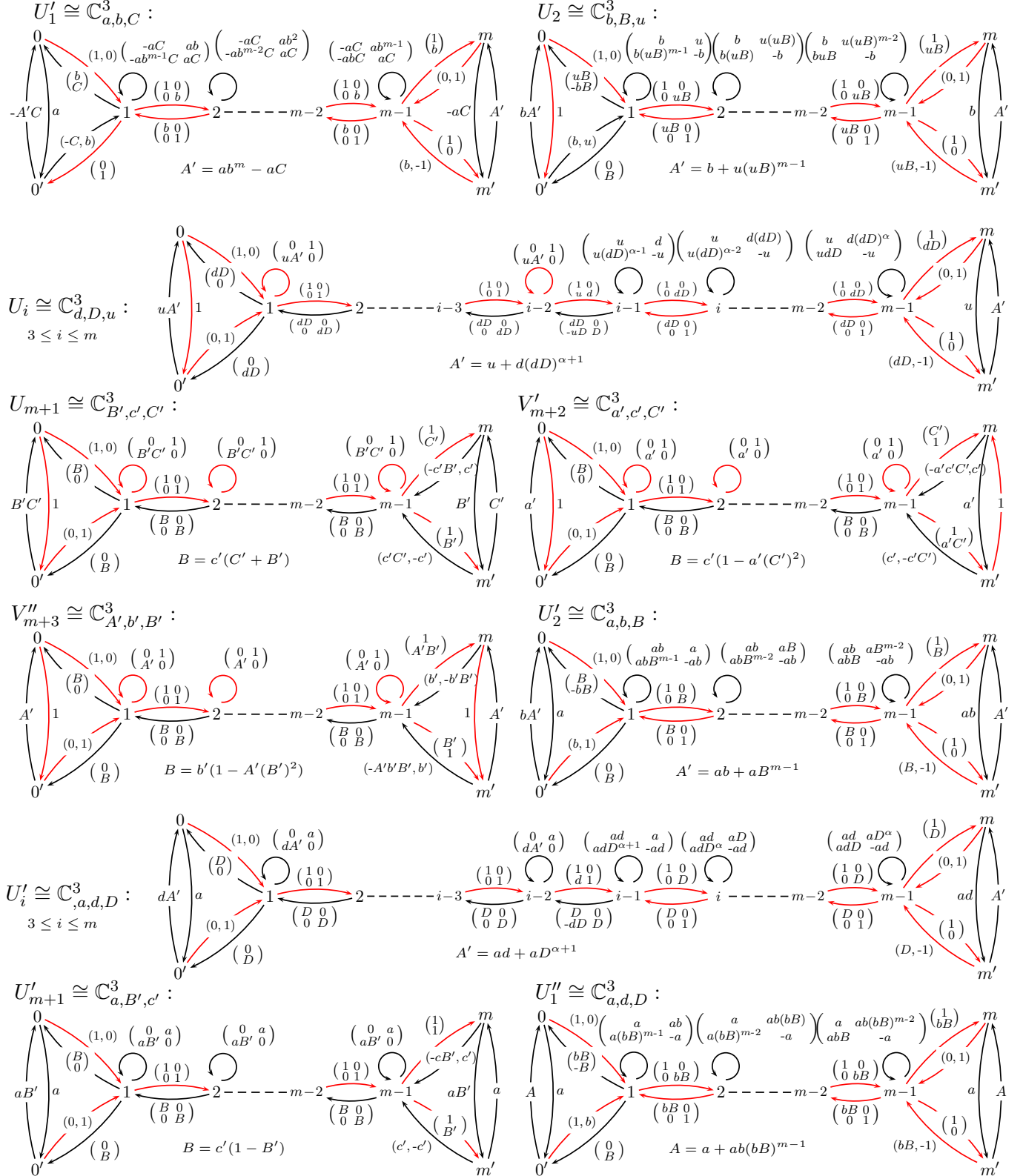
FIGURE 6. Open sets for crepant resolutions of \mathbb{C}^3/G for $G = D_{2n}$, n even (I).

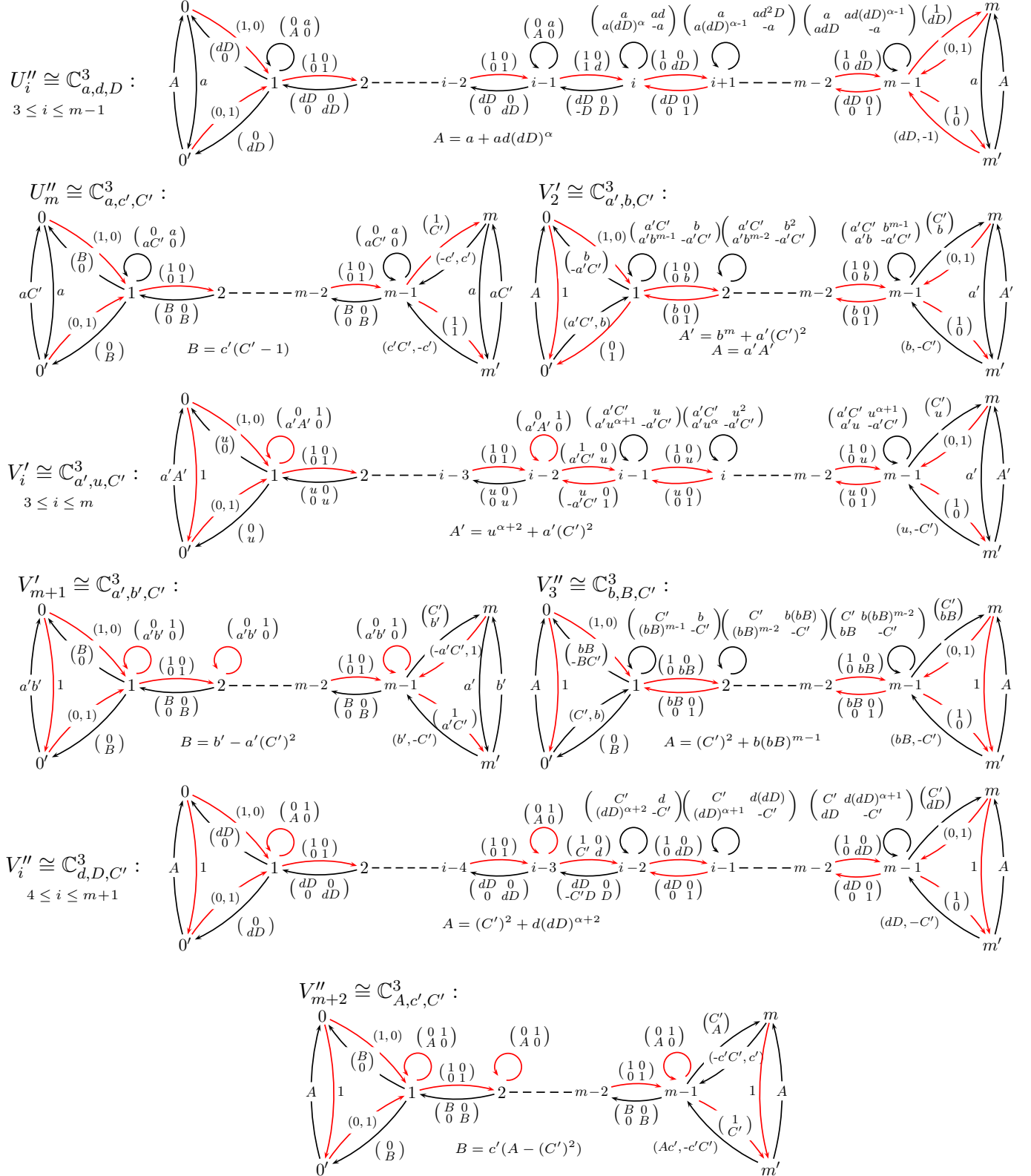
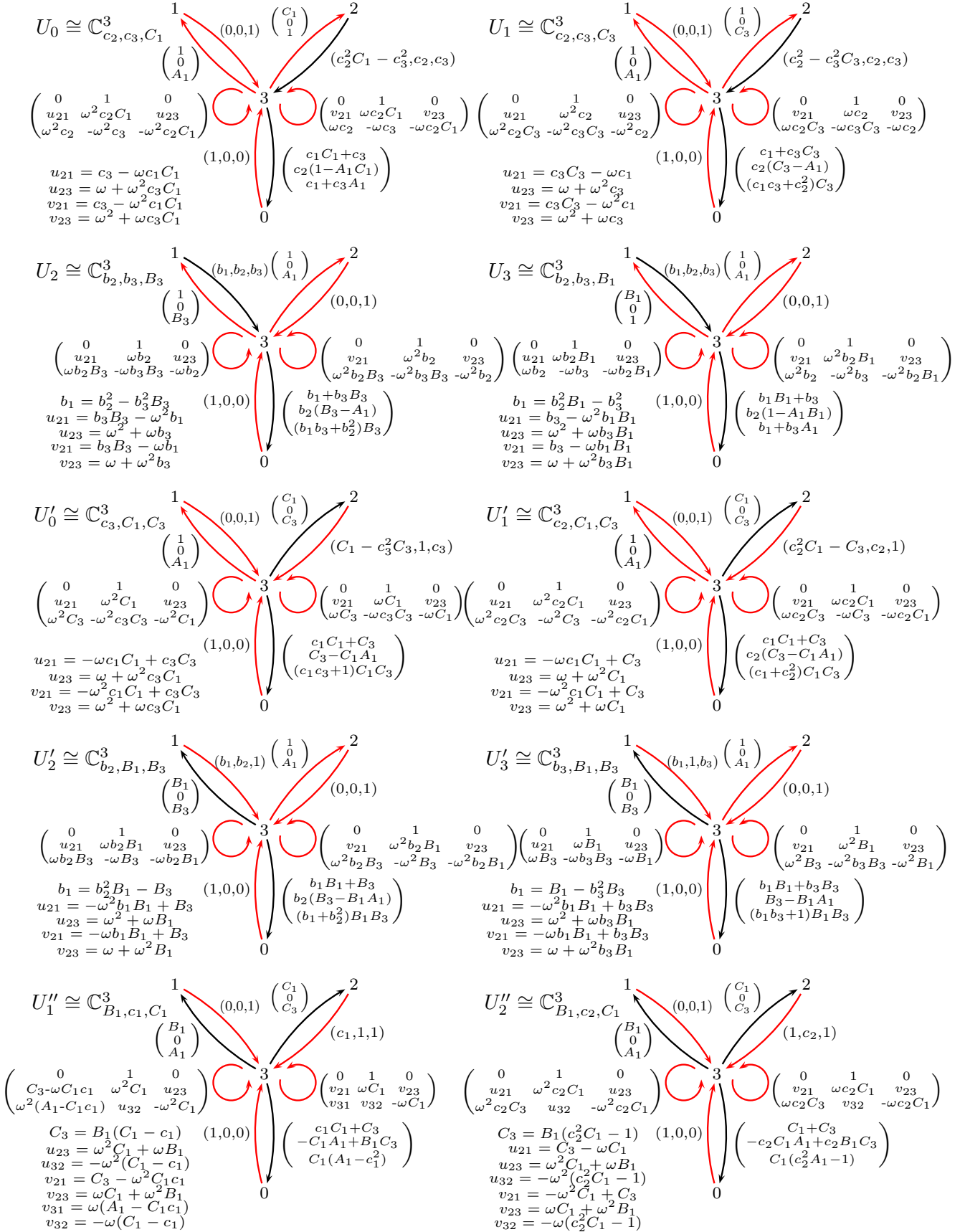
FIGURE 7. Open sets for crepant resolutions of \mathbb{C}^3/G for $G = D_{2n}$, n even (II).

FIGURE 8. Open sets for crepant resolutions of \mathbb{C}^3/G for $G = \mathbb{T} \subset SO(3)$.

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